IDEAL GLM-MHD

ENTROPY CONSISTENT IDEAL MHD WITH INBUILT DIVERGENCE CLEANING

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A NEW MODEL - WHY?

Ideal MHD solely derived from conservation laws:

$$\varrho,\ \varrho\vec{u},\ E,\ \vec{B}$$

Essential physics is missing:

- 2^{nd} law of TD: Entropy inequality
- Divergence-free condition ($\nabla \cdot \vec{B} = 0$)

A NEW MODEL - HOW?

1. Introduce entropy as a new quantity
2. Entropy-conserving (EC) scheme (smooth flows)
$$S_t + \nabla(\vec{u}S) = 0$$
3. Entropy-stable (ES) scheme (arbitrary flows)
$$S_t + \nabla(\vec{u}S) \leq 0$$
4. Add divergence constraint

IDEAL MHD

$$\frac{\partial}{\partial t}\vec{q} + \nabla\cdot\vec{f} = \frac{\partial}{\partial t}\begin{bmatrix}\varrho \\ \varrho\vec{u} \\ E \\ \vec{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\varrho\vec{u} \\ \varrho(\vec{u}\otimes\vec{u}) + \left(p + \frac{1}{2}\|\vec{B}\|^2\right)\mathbf{I} - \vec{B}\otimes\vec{B} \\ \vec{u}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{u}\cdot\vec{B}) \\ \vec{u}\otimes\vec{B} - \vec{B}\otimes\vec{u}\end{bmatrix} = \vec{0}, \tag{1}$$

$$\nabla\cdot\vec{B} = 0$$

Not Galilean invariant ↯

IDEAL MHD - DIVERGENCE-FREE CONSTRAINT

Essential assumption: $\nabla \cdot \vec{B} = 0$

Numerical reality: $\nabla \cdot \vec{B} \approx 0$

ideal MHD: multiple issues, e.g. wrong Lorentz force:

$$\vec{F}_L \cdot \frac{\vec{B}}{\|\vec{B}\|} = -(\nabla \cdot \vec{B})\|\vec{B}\| \neq 0$$

$\rightarrow$ artificial force $\parallel \vec{B}$ ↯

Ideal MHD equations are invalid (!) for $\nabla \cdot \vec{B} \approx 0$

IDEAL MHD - ENTROPY-CONSISTENCY

2^{nd} law of TD:

$$\vec{v}^{\mathsf{T}}(\vec{q}_t + \nabla \cdot \vec{f}) = S_t + \nabla(\vec{u}S) \leq 0$$

$$\vec{v}^{\mathsf{T}}\vec{q}_t = S_t$$

$$\vec{v}^{\mathsf{T}}(\nabla \cdot f_{\text{hydro}}) = \nabla(\vec{u}S)$$

$$\vec{v}^{\mathsf{T}}(\nabla \cdot f_{\text{magnetic}}) = (\nabla \cdot \vec{B})(\vec{u} \cdot \vec{B}) \neq 0$$

Not entropy consistent for $\nabla \cdot \vec{B} \neq 0$ ↯

IDEAL MHD: DERIVATION

- Start from Euler equations

$$\frac{\partial}{\partial t}\vec{q} + \nabla\cdot\vec{f} = \frac{\partial}{\partial t}\begin{bmatrix}\varrho\\ \varrho\vec{u}\\ E\end{bmatrix} + \nabla\cdot\begin{bmatrix}\varrho\vec{u}\\ \varrho(\vec{u}\otimes\vec{u}) + p\mathbf{I}\\ \vec{u}\big(E+p\big)\end{bmatrix} = \begin{bmatrix}0\\ \vec{F}\\ E_a\end{bmatrix}$$

- Use Maxwell's equations (induction eq.)
- Add Lorentz force (momentum coupling)

- Interesting: Result is already known (Godunov, 1972)

$\longrightarrow$ eight-wave formulation

IDEAL MHD (8 WAVE FORMULATION)

$$\frac{\partial}{\partial t}\vec{q} + \nabla \cdot \vec{f} = \frac{\partial}{\partial t}\begin{bmatrix} \varrho \\ \varrho\vec{u} \\ E \\ \vec{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \varrho\vec{u} \\ \varrho(\vec{u}\otimes\vec{u}) + \left(p + \frac{1}{2}\|\vec{B}\|^2\right)\mathbf{I} - \vec{B}\otimes\vec{B} \\ \vec{u}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{u}\cdot\vec{B}) \\ \vec{u}\otimes\vec{B} - \vec{B}\otimes\vec{u} \end{bmatrix} = -(\nabla\cdot\vec{B})\begin{bmatrix} 0 \\ \vec{B} \\ \vec{u}\cdot\vec{B} \\ \vec{u} \end{bmatrix} \tag{2}$$

Galilean invariant ✓

8 WAVE - DIVERGENCE-FREE CONSTRAINT

Assumption: $\nabla \cdot \vec{B} \approx 0$

$$\vec{F}_L \cdot \frac{\vec{B}}{\|\vec{B}\|} = 0$$

$\longrightarrow$ *no* artificial force for *any* $\nabla \cdot \vec{B}$

8 WAVE - ENTROPY-CONSISTENCY

2^{nd} law of TD:

$$\vec{v}^{\intercal}(\vec{q}_t + \nabla \cdot \vec{f} - \vec{\Upsilon}) = S_t + \nabla(\vec{u}S) \leq 0$$

$$\vec{v}^{\intercal}\vec{q}_t = S_t$$

$$\vec{v}^{\intercal}(\nabla \cdot f_{\text{hydro}}) = \nabla(\vec{u}S)$$

$$\vec{v}^{\intercal}(\nabla \cdot f_{\text{magnetic}} - \vec{\Upsilon}) = 0$$

Entropy consistent ✓

(Derigs et al., 2016 + 2017)

What about $|\nabla \cdot \vec{B}|$?

THE DIVERGENCE-FREE PROBLEM (1)

$\nabla \cdot \vec{B} \to 0$: Important for accurate and correct solutions

However, 8 wave formulation leaves $\nabla \cdot \vec{B}$ (almost) uncontrolled

Solution: Divergence “cleaning” methods

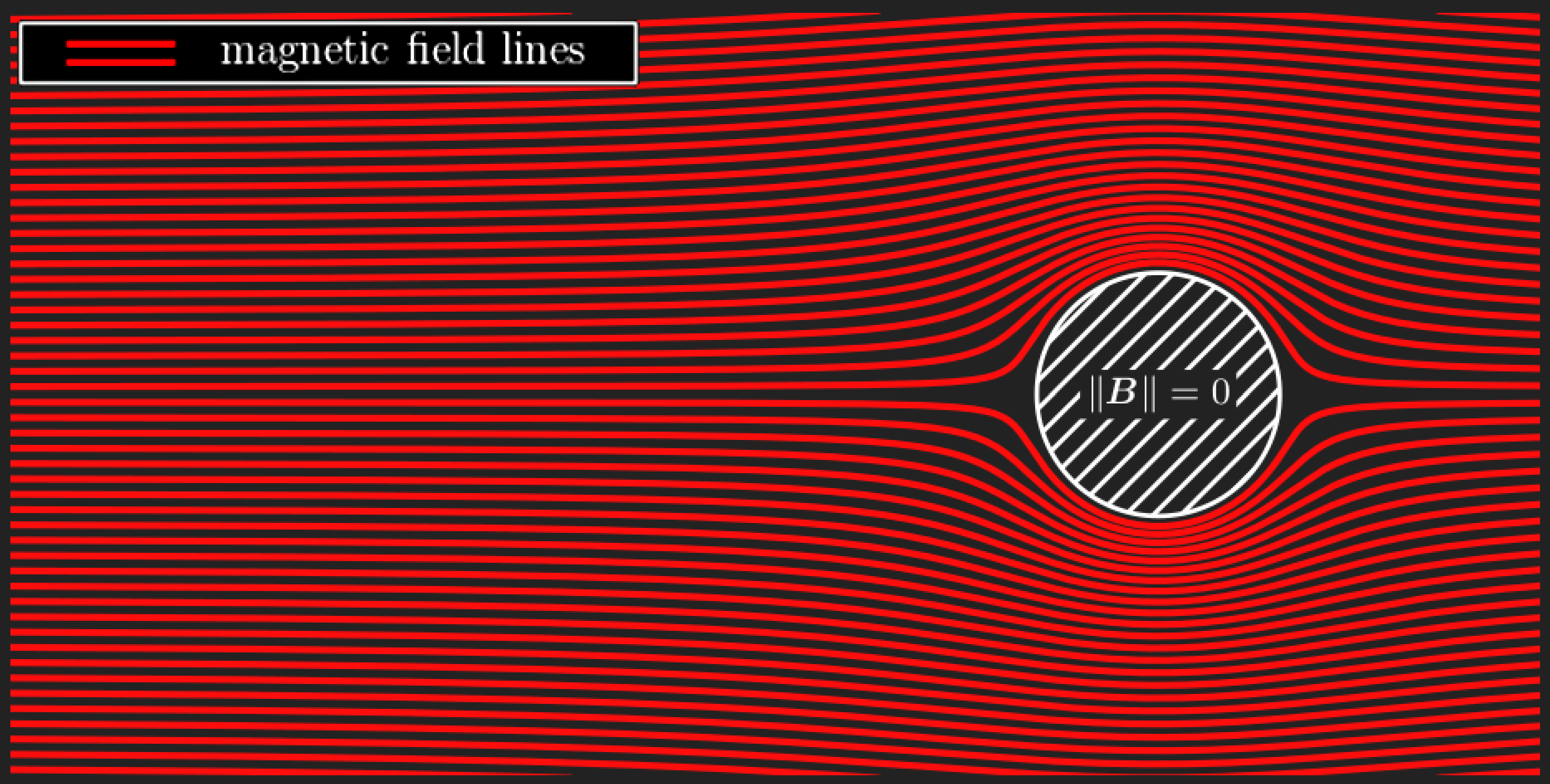
THE DIVERGENCE-FREE PROBLEM (2)

Available methods:

- Source term (Godunov, Powell et al., Janhunen et al.) ↯
- Projection method (Brackbill and Barnes) ↯
- Diffusion method (van der Holst and Keppens) ↯
- Constrained transport (Evans and Hawley) ↯
- GLM formulation (Munz et al., Dedner et al.) ✓

GLM - WHY?

Task: Minimize $\nabla \cdot \vec{B}$



Change in magnetic field: non-local

Effect of magnetic field: purely local

Propagate local changes through the grid

GLM - HOW?

Step 1: Add GLM minimization for $\nabla \cdot \vec{B}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi + \nabla(c_h \vec{B}) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{B} + \nabla(\vec{u} \otimes \vec{B} - \vec{B} \otimes \vec{u}) + \nabla(c_h \psi) = 0$$

ψ field propagates $\vec{B}$ changes outwards

$\longrightarrow$ Step 2: Add new energy in ψ field

$$E = \frac{1}{2}\varrho\|\vec{u}\|^2 + \frac{1}{2}\|\vec{B}\|^2 + \frac{1}{2}\psi^2$$

$\longrightarrow$ Step 3: Modify total energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{E} + \nabla\left(\vec{u}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{u} \cdot \vec{B}) + c_h \psi \vec{B}\right) = \dots$$

8 WAVE + GLM = 9 WAVE

$$+ \nabla \cdot \vec{f} = \frac{\partial}{\partial t}\begin{bmatrix} \varrho \\ \varrho\vec{u} \\ E \\ \vec{B} \\ \psi \end{bmatrix} + \nabla \cdot \begin{bmatrix} \varrho\vec{u} \\ \varrho(\vec{u}\otimes\vec{u}) + \left(p + \frac{1}{2}\|\vec{B}\|^2\right)\mathbf{I} - \vec{B}\otimes\vec{B} \\ \vec{u}\left(E + p + \frac{1}{2}\|\vec{B}\|^2\right) - \vec{B}(\vec{u}\cdot\vec{B}) + c_h\psi\vec{B} \\ \vec{u}\otimes\vec{B} - \vec{B}\otimes\vec{u} + c_h\psi \\ c_h\vec{B} \end{bmatrix} = -(\nabla\cdot\vec{B})\begin{bmatrix} 0 \\ \vec{B} \\ \vec{u}\cdot\vec{B} \\ \vec{u} \\ 0 \end{bmatrix} - (\nabla\psi)\begin{bmatrix} 0 \\ 0 \\ \vec{u}\psi \\ 0 \\ \vec{u} \end{bmatrix}, \tag{3}$$

Galilean invariant ✓

$\nabla \cdot \vec{B}$ diminishing ✓

thermodynamically consistent ✓

9 WAVE = ENTROPY-CONSISTENT

2^{nd} law of TD:

$$\vec{v}^{\intercal}(\vec{q}_t + \nabla \cdot \vec{f} - \vec{\Upsilon}) = S_t + \nabla(\vec{u}S) \leq 0$$

$$\begin{aligned} \vec{v}^{\intercal}\vec{q}_t &= S_t \\ \vec{v}^{\intercal}(\nabla \cdot f_{\text{hydro}}) &= \nabla(\vec{u}S) \\ \vec{v}^{\intercal}(\nabla \cdot f_{\text{magnetic}} - \vec{\Upsilon}_{\text{mag}}) &= 0 \\ \vec{v}^{\intercal}(\nabla \cdot f_{\text{GLM}} - \vec{\Upsilon}_{\text{GLM}}) &= 0 \end{aligned}$$

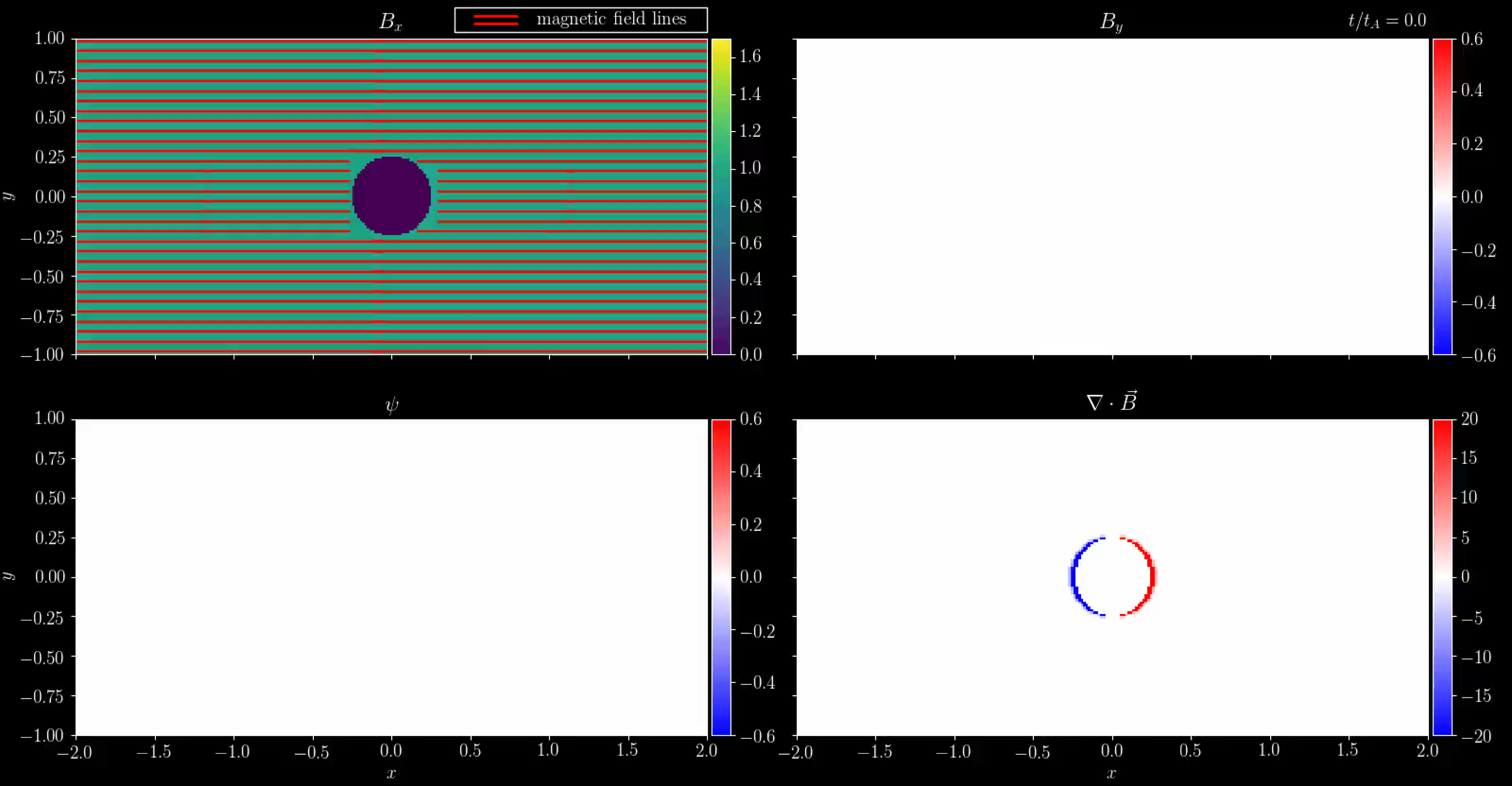
Entropy consistent ✓

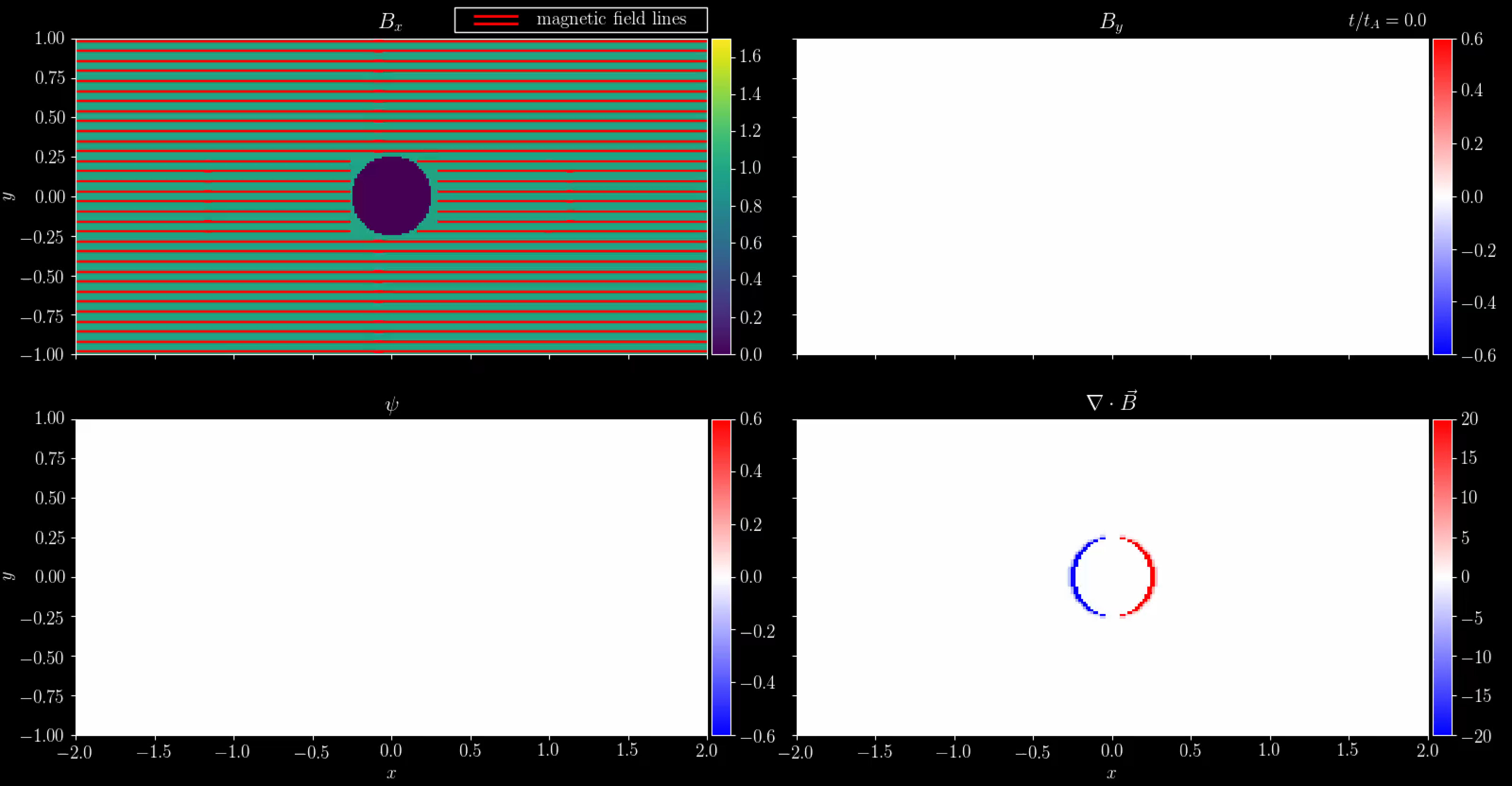
NUMERICAL RESULTS

- MHD “Superconductor” test
- Orszag-Tang MHD vortex test

- Resolution: AMR up to 256×256
- Reference solution: `FLASH`'s USM solver

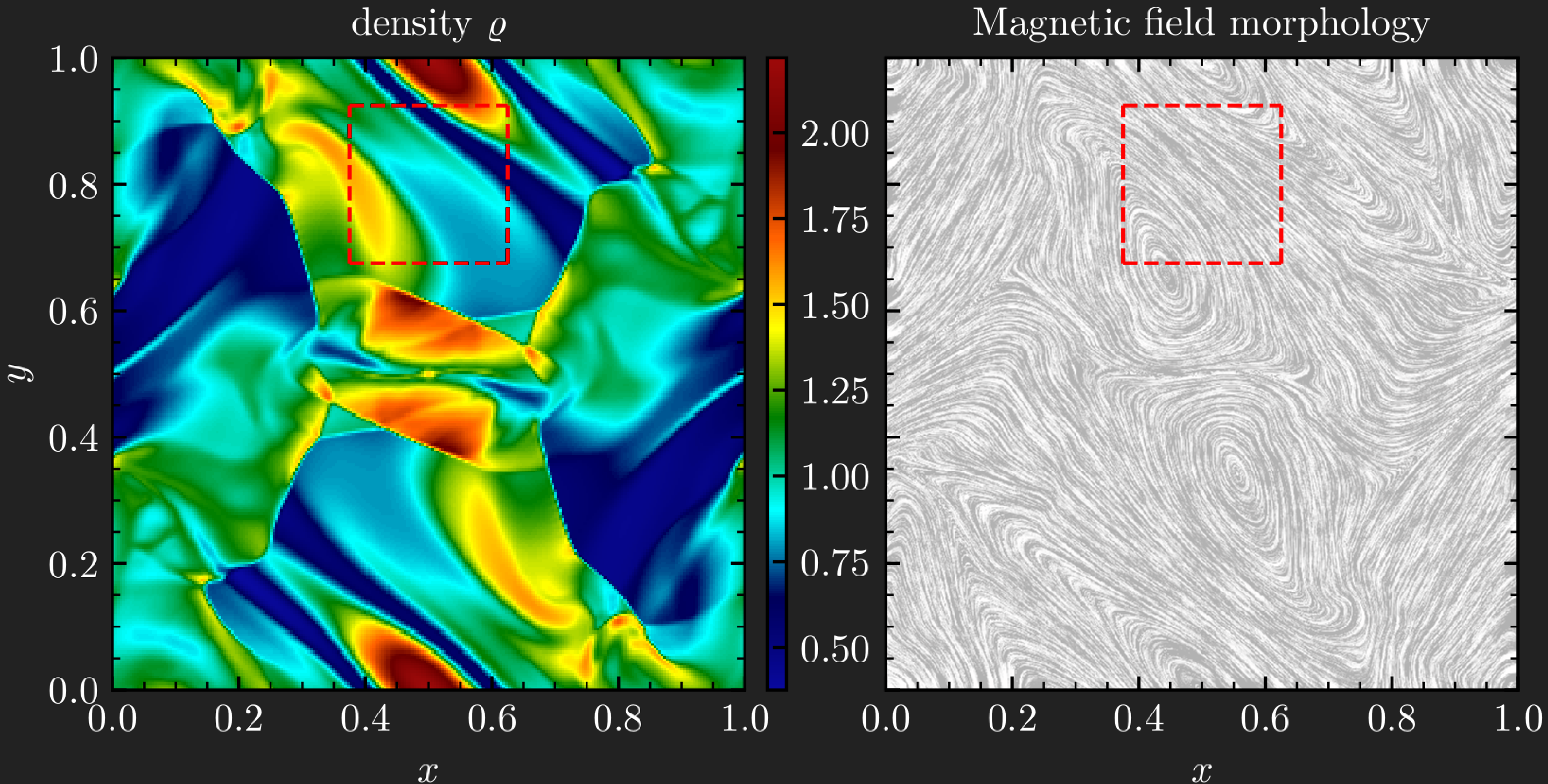
MHD “SUPERCONDUCTOR” TEST


B_x
magnetic field lines
B_y
$t/t_A = 0.0$
ψ
$\nabla \cdot \vec{B}$
y
x

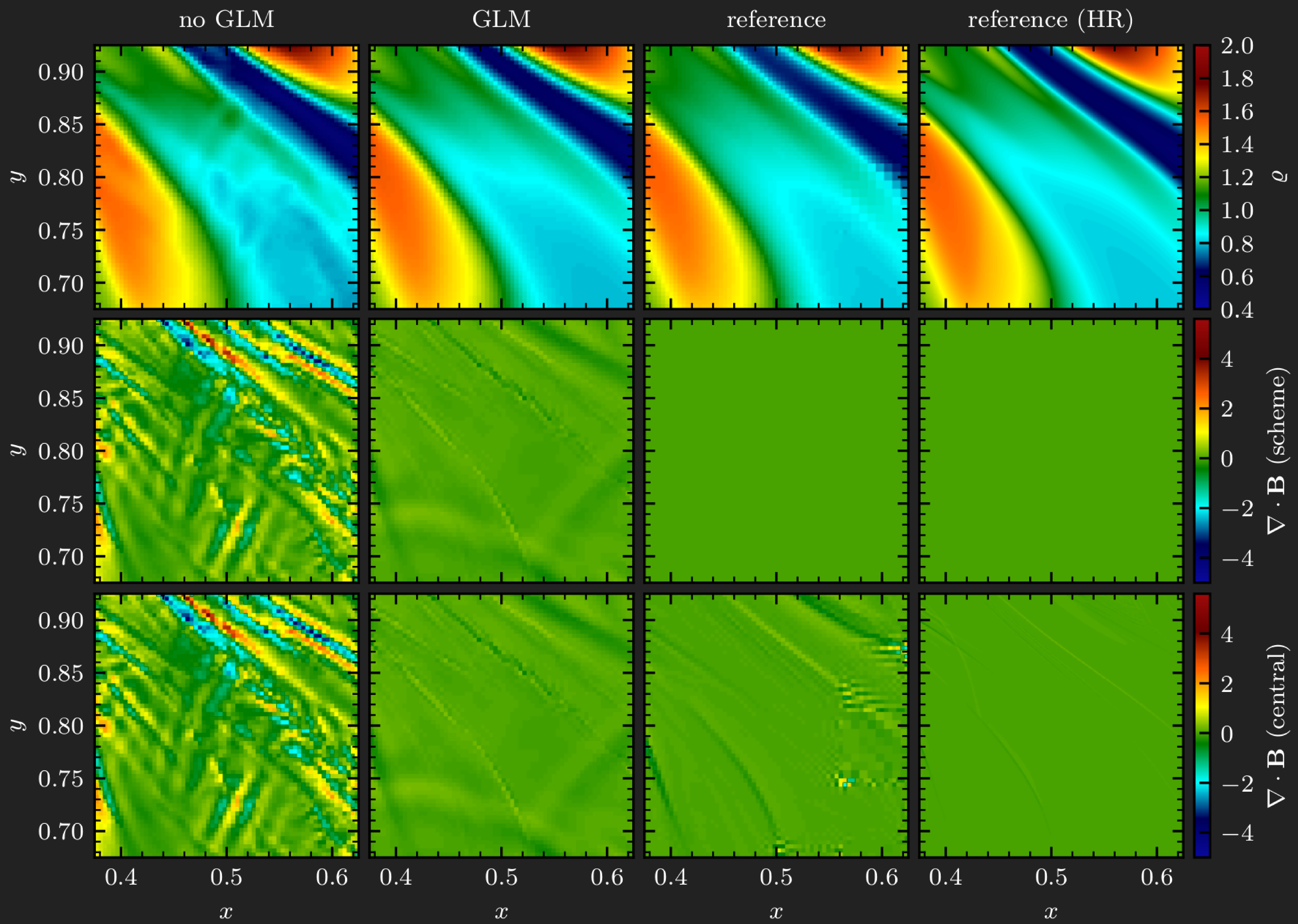


B_x
magnetic field lines
B_y
$t/t_A = 0.0$
ψ
$\nabla \cdot \vec{B}$
x
y

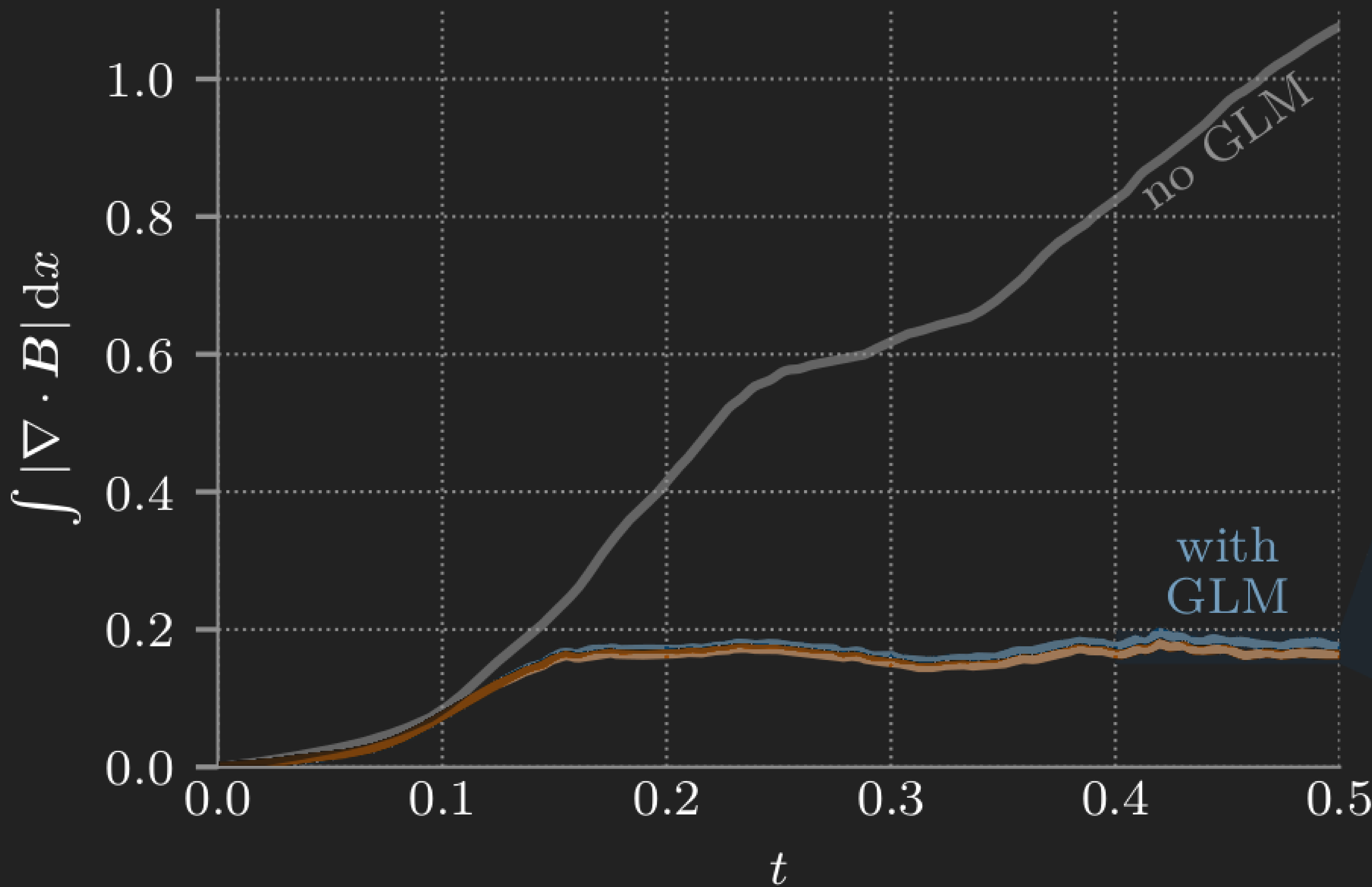
ORSZAG-TANG VORTEX TEST



density ϱ
Magnetic field morphology
1.0
0.8
0.6
0.4
0.2
0.0
y
x
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.0
0.2
0.4
0.6
0.8
1.0



no GLM
GLM
reference
reference (HR)
0.90
0.85
0.80
0.75
0.70
y
0.4
0.5
0.6
x
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
ϱ
4
2
0
−2
−4
∇ · B (scheme)
∇ · B (central)



no GLM
with
GLM
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
$\int |\nabla \cdot \boldsymbol{B}| \, \mathrm{d}x$
t

SUMMARY

I obtained the first Riemann solver that is

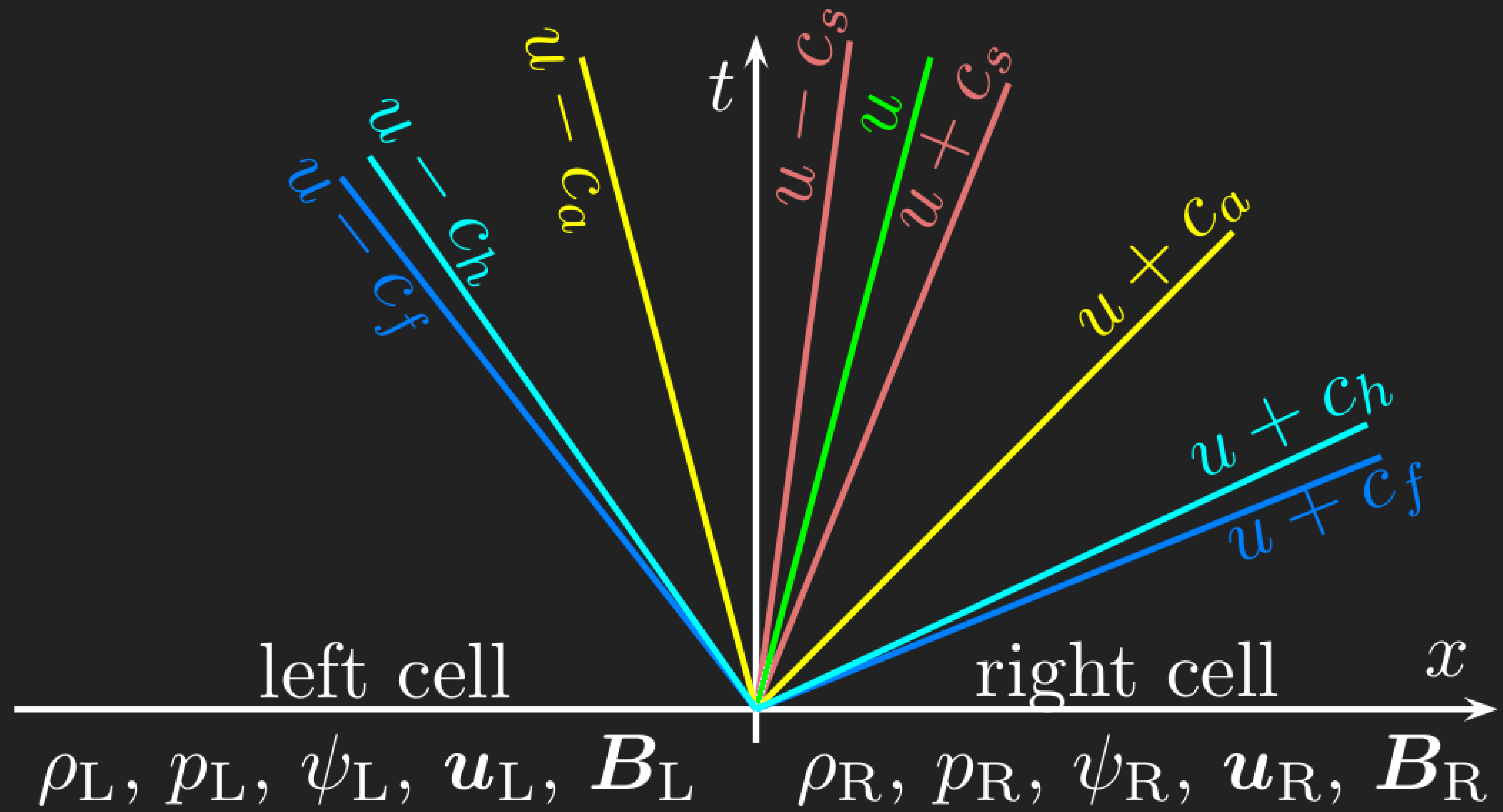
- $\nabla \cdot \vec{B}$ diminishing by construction,
- thermodynamically consistent,
- (fairly) simple,
- computationally efficient,
- fully compatible with techniques like AMR, and
- arbitrarily accurate.

Things I have not talked about:

- kinetic energy preserving properties of my Riemann solver,
- entropy properties of other implementations of GLM,
- conservative GLM implementation,
- mixed GLM (active ψ damping),
- technical details of the ES scheme (derivations, etc.)

$\longrightarrow$ ARXIV: 1711.06269 $\longleftarrow$

9 WAVE: WAVESPEEDS



$$\lambda^x_{\pm \rm f} = u \pm c_{\rm f}, \qquad \lambda^x_{\pm \psi} = u \pm c_h, \qquad \lambda^x_{\pm \rm a} = u \pm c_{\rm a}, \qquad \lambda^x_{\pm \rm s} = u \pm c_{\rm s}, \qquad \lambda^x_{\rm E} = u$$