Monte Carlo Tracer Particles for non-Lagrangian codes

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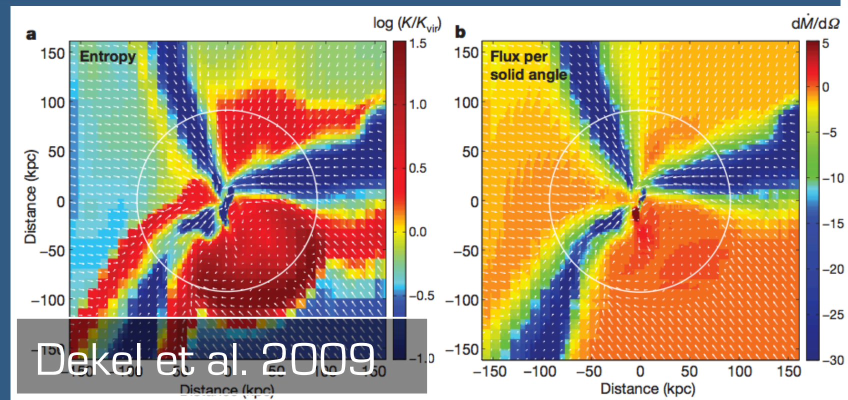
Outline

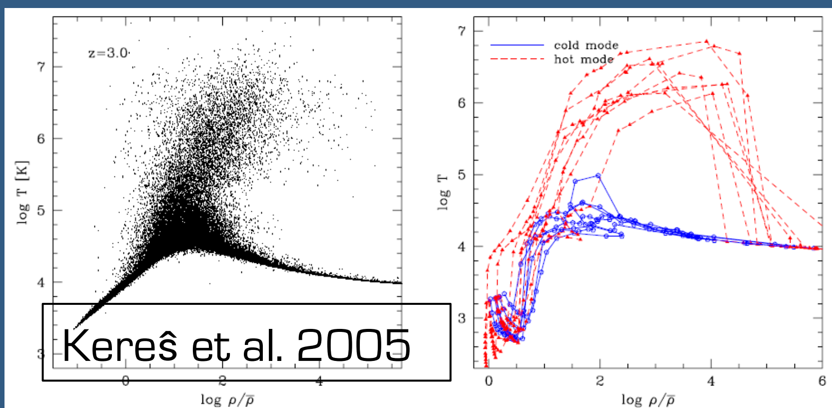
- A brief motivation for Lagrangian analysis in galaxy formation
- Velocity Field tracer particles: good or bad?
- Monte Carlo tracer particles: an independent approach
- Directional Monte Carlo tracer particles: on-going work
- When are tracer particles necessary? What defines a numerical method as ‘Lagrangian’?

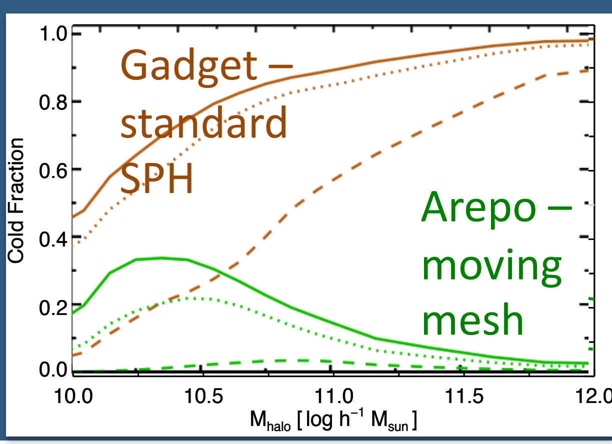
Cosmological gas accretion

Does gas shock-heat before accreting onto galaxies? How much and where?

Still pictures aren't enough!







Nelson et al. 2013, 2015:

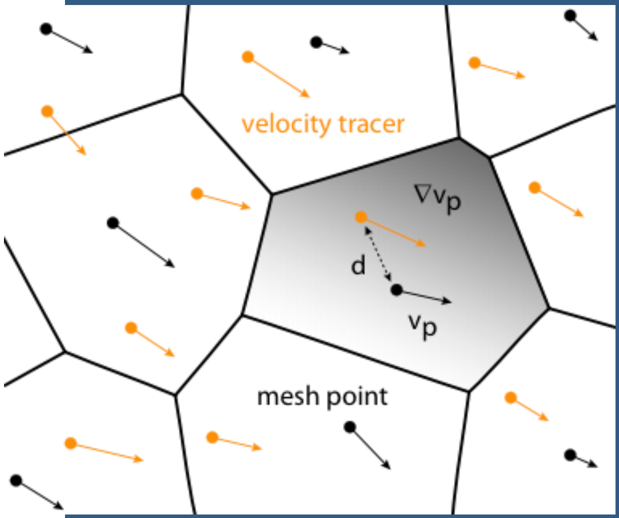
First (and still only) Lagrangian analysis of cosmological accretion with a non-SPH code

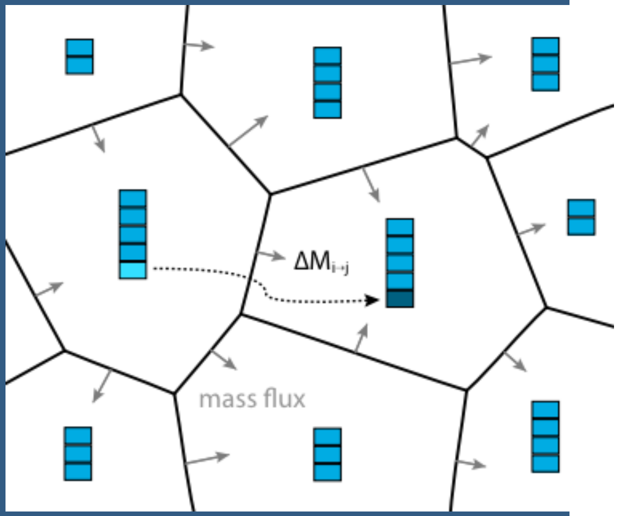
Lagrangian tracer particles

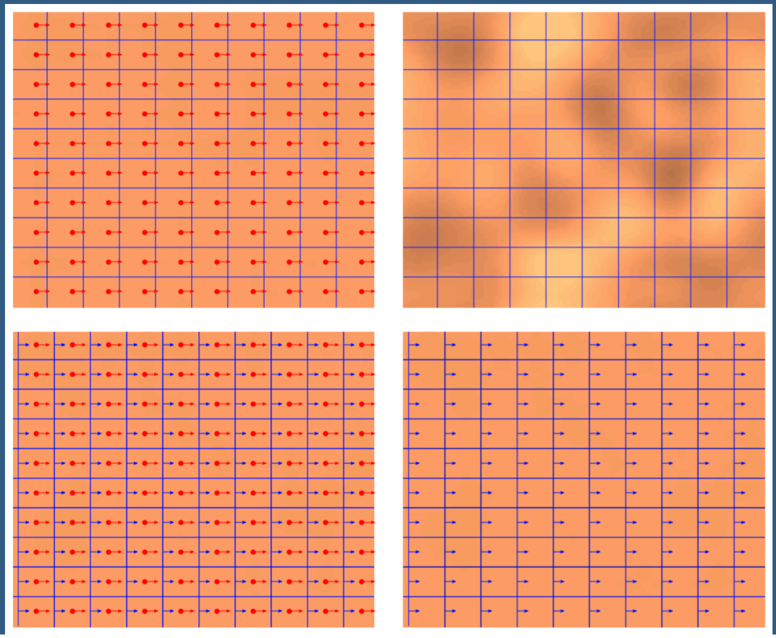
Genel et al. 2013

Velocity field tracers

Monte Carlo tracers







Red dots – velocity field tracers

Fixed mesh

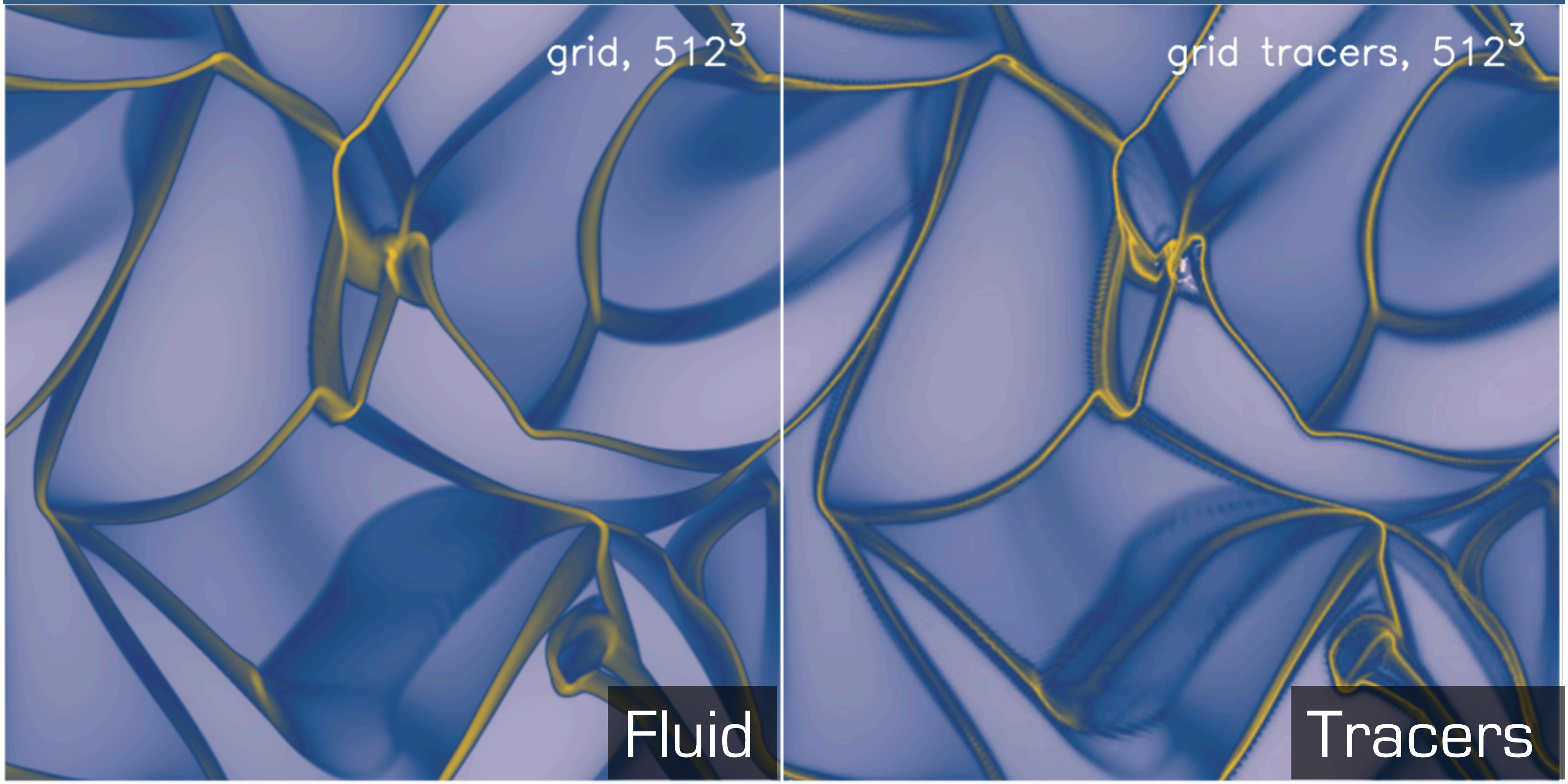
Background – tracer density

Moving mesh

Price & Federrath 2010

Velocity Field tracers

Supersonic (M≅10) isothermal hydro turbulence (solenoidal driving), using FLASH (AMR), after 1 dynamical time



Velocity Field tracers

Price & Federrath 2010

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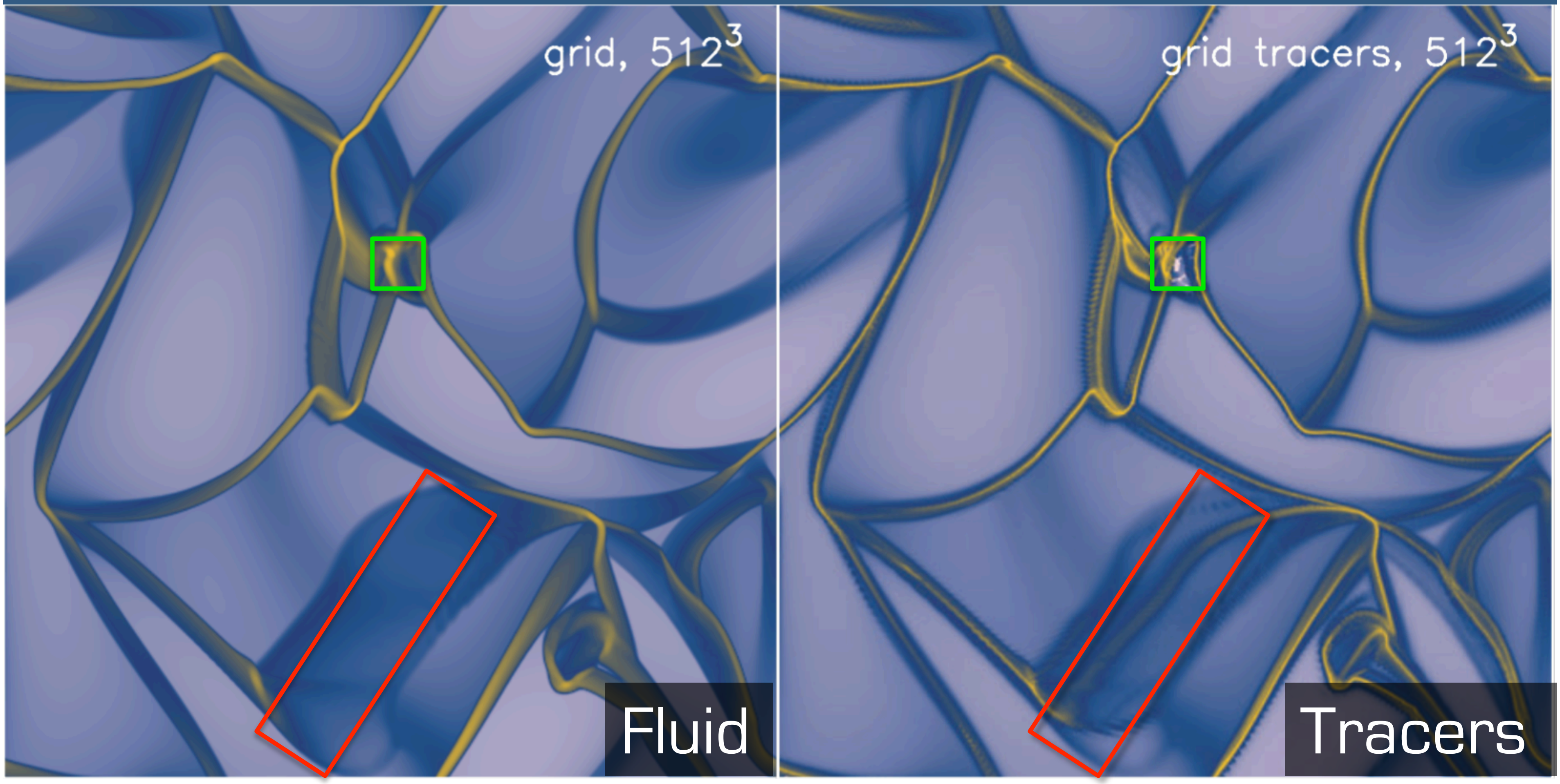
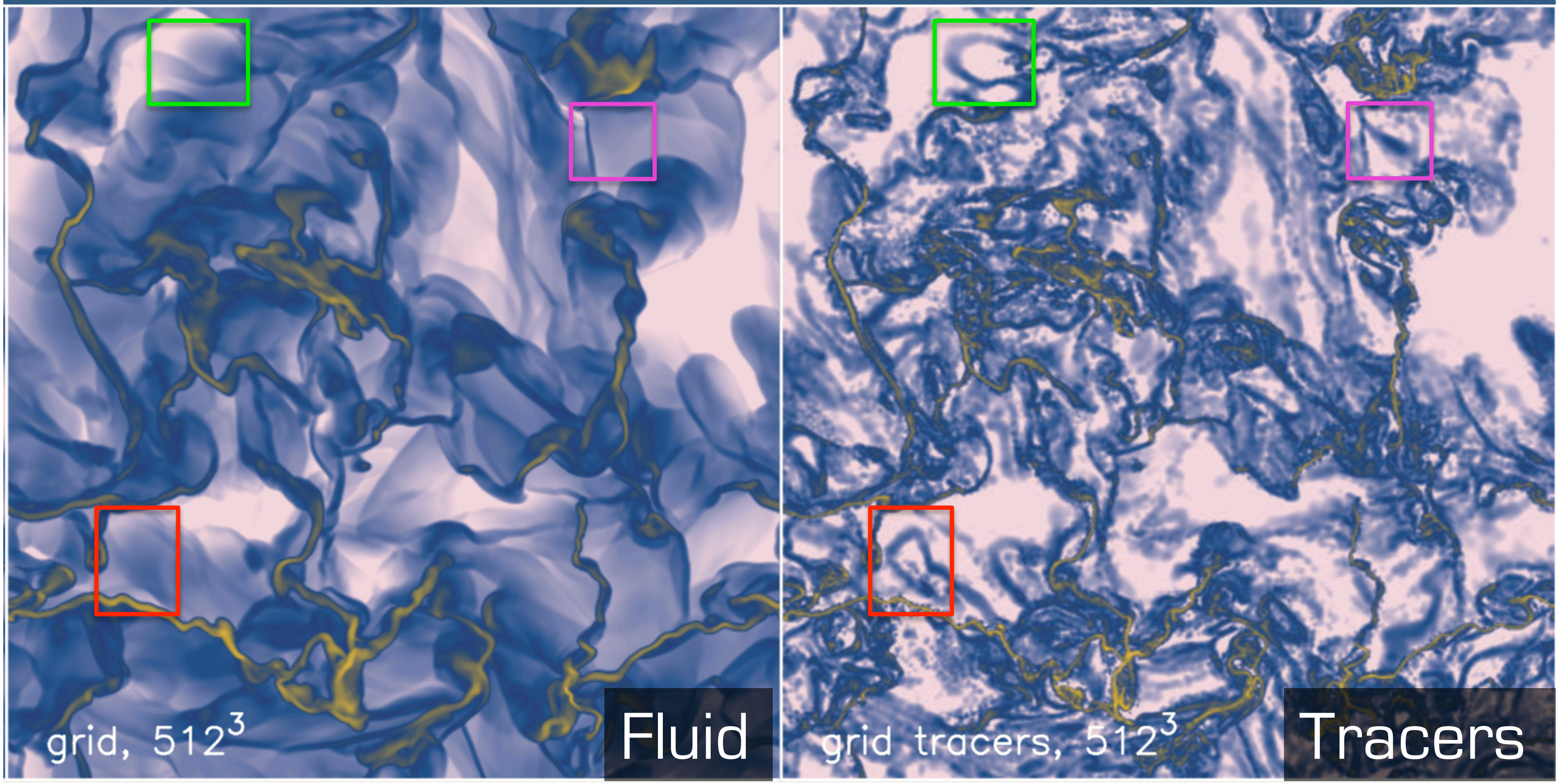


Price & Federrath 2010

Velocity Field tracers

Supersonic (M≅10) isothermal hydro turbulence (solenoidal driving), using FLASH (AMR), after 10 dynamical times

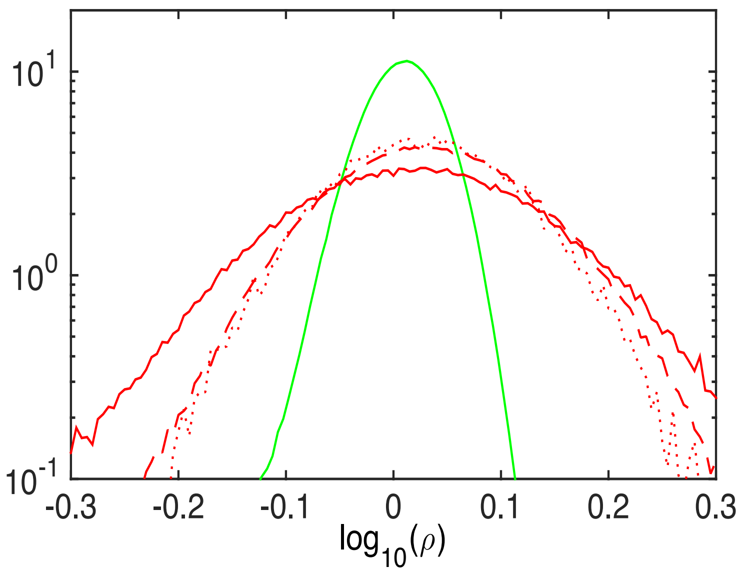


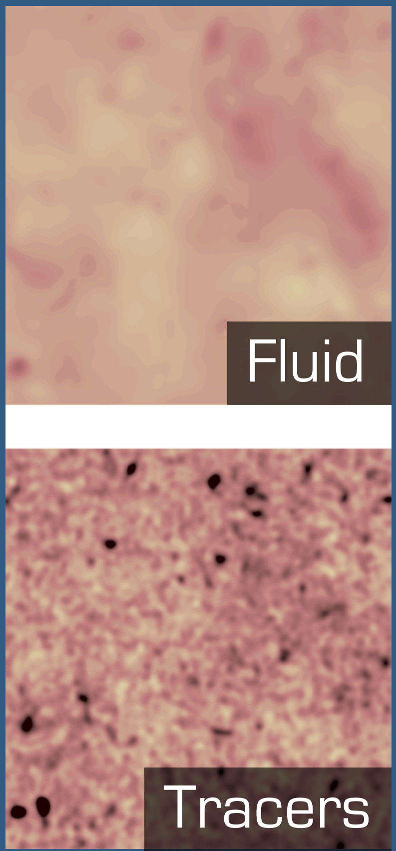
Genel et al. 2013

Velocity Field tracers

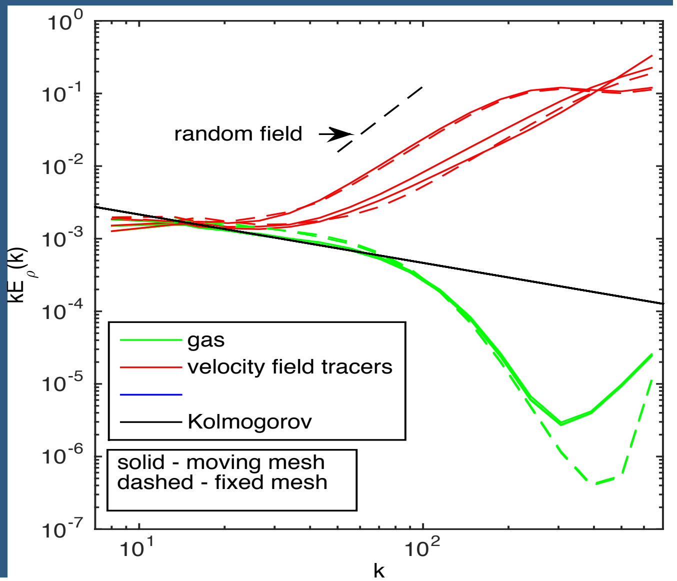
Subsonic (M≅0.3) isothermal hydro turbulence (solenoidal driving), using Arepo (moving-mesh), in steady state

Density PDF broadened with respect to the fluid





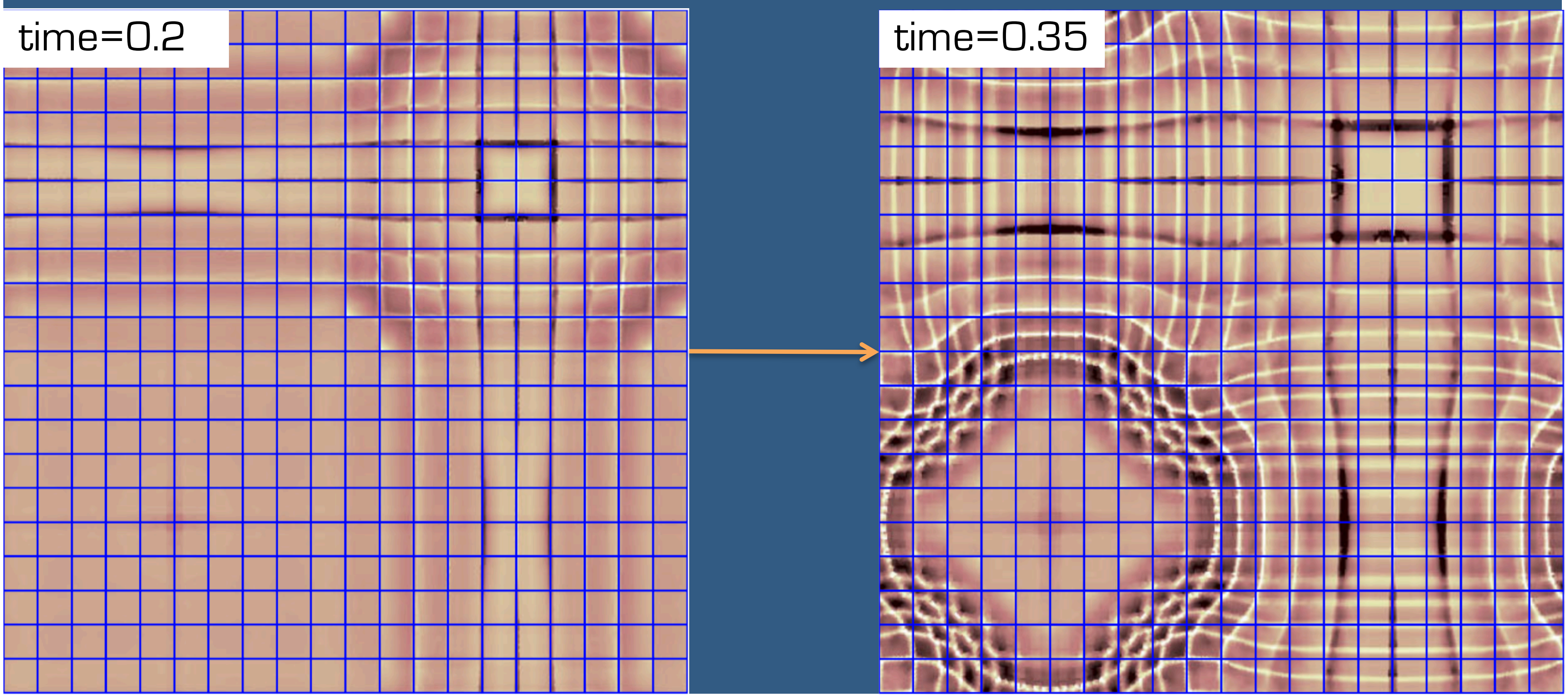
Density power spectrum dominated by noise



Genel et al. 2013

Velocity Field tracers

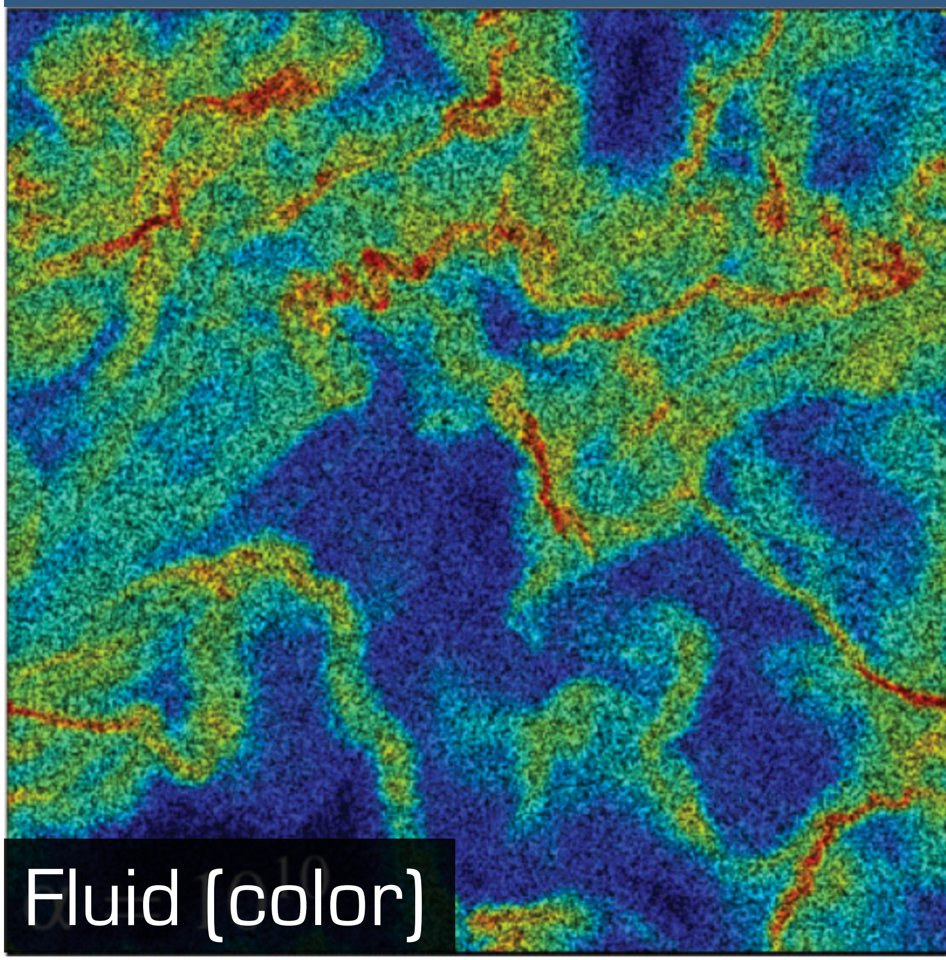
- Tracer over-/under-densities develop in particular co-spatially with discontinuities
- They are advected, 'forgetting' their formation sites

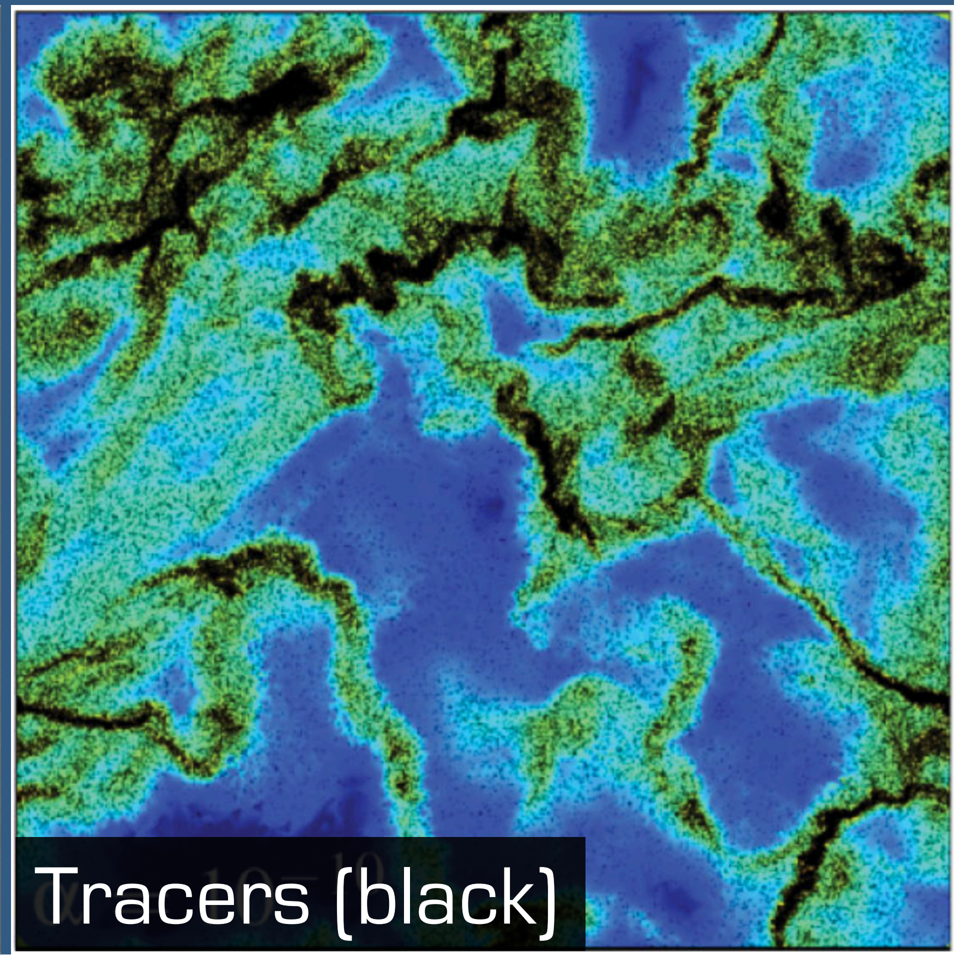


Hopkins & Lee 2016

Velocity Field tracers

Supersonic (M≅10) isothermal MHD turbulence (solenoidal driving), using GIZMO (MFM/MFV), in steady state

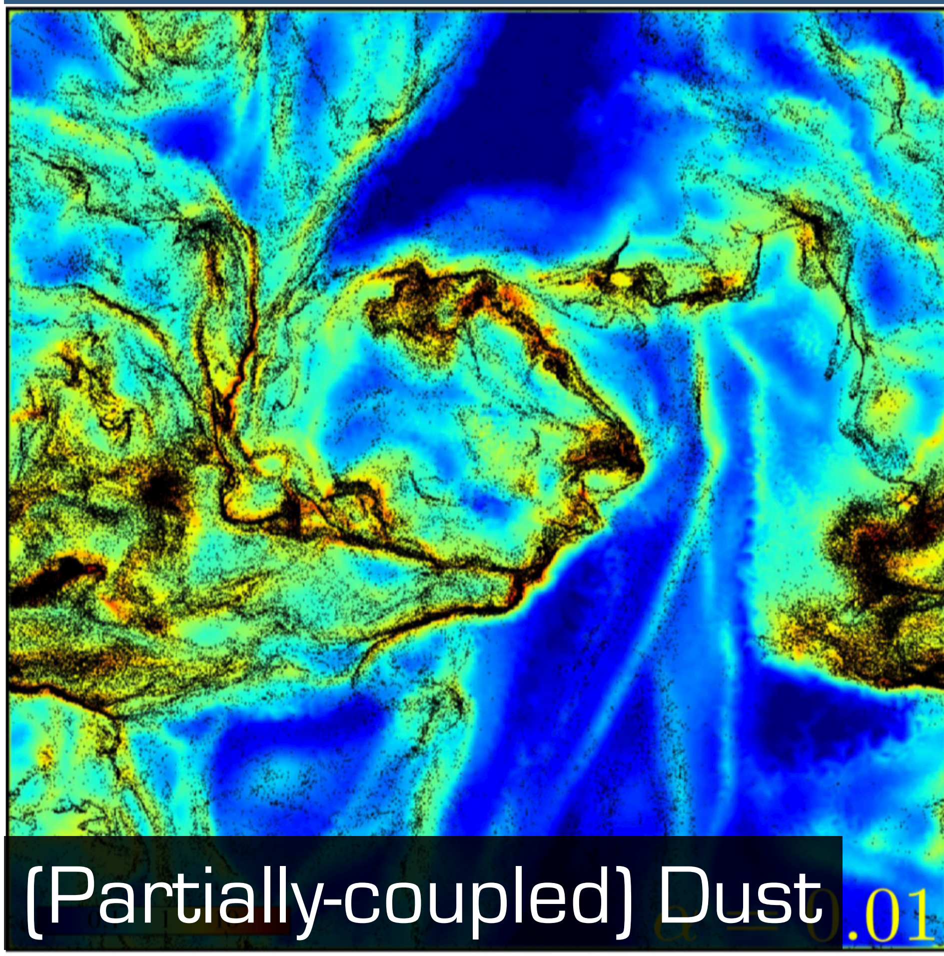




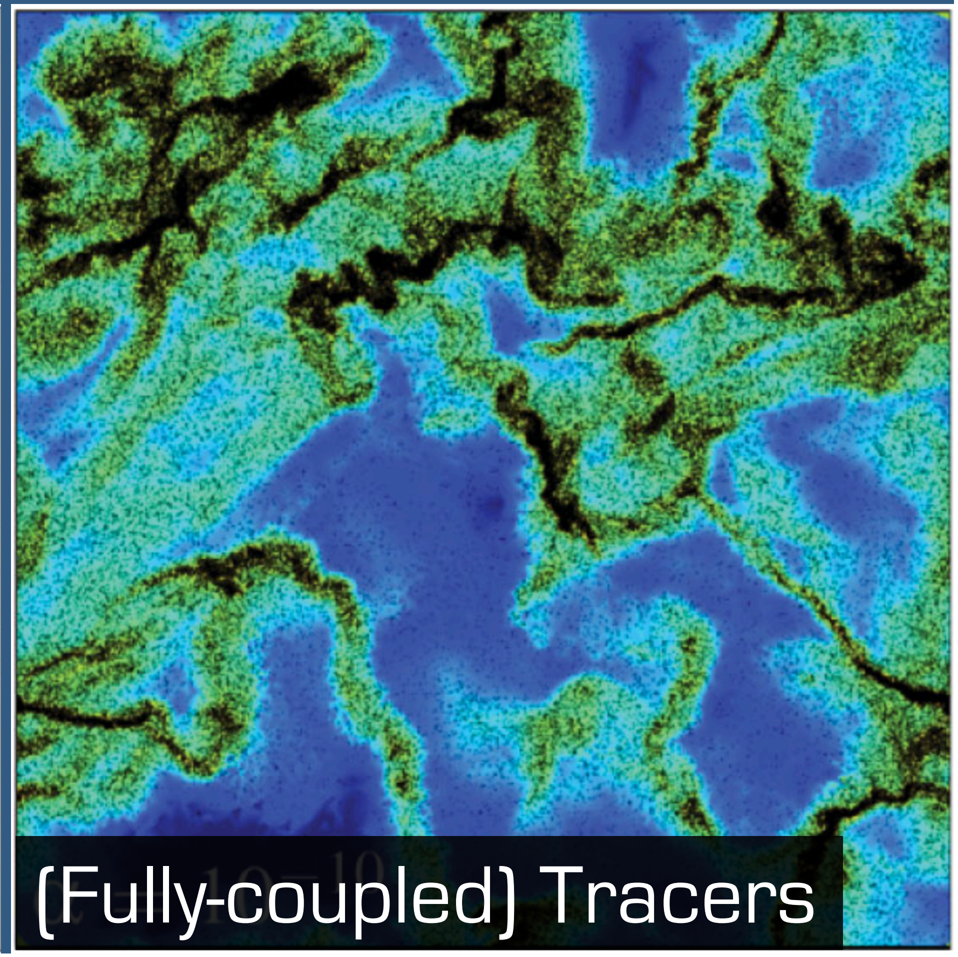
Hopkins & Lee 2016

Velocity Field tracers as dust

Supersonic ($\mathcal{M} \cong 10$) isothermal MHD turbulence (solenoidal driving), using GIZMO (MFM/MFV), in steady state



(Partially-coupled) Dust



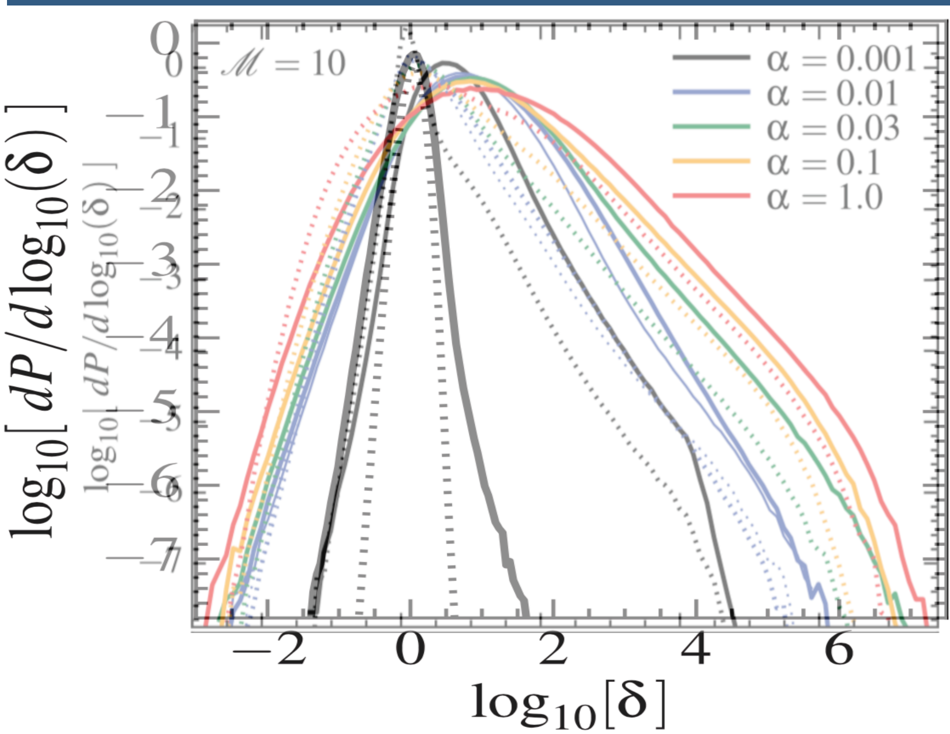
(Fully-coupled) Tracers

Hopkins & Lee 2016

Velocity Field tracers as dust

Supersonic (M≅10) isothermal MHD turbulence (solenoidal driving), using GIZMO (MFM/MFV), in steady state

Fully-coupled tracers still show ~0.1dex deviations from fluid, but this is small compared to physical effects of partially-coupled dust



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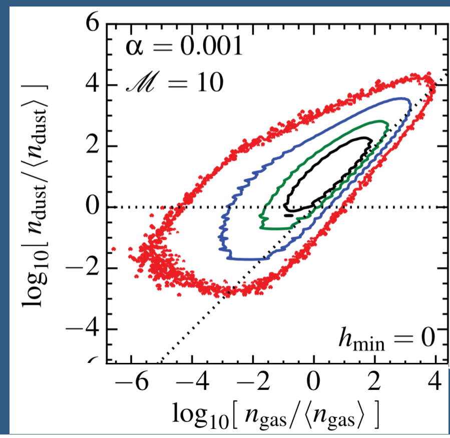
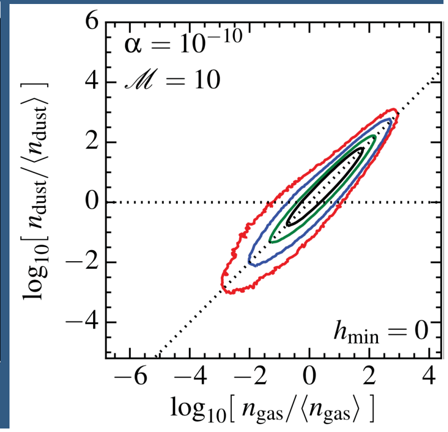


(Fully-coupled) Tracers

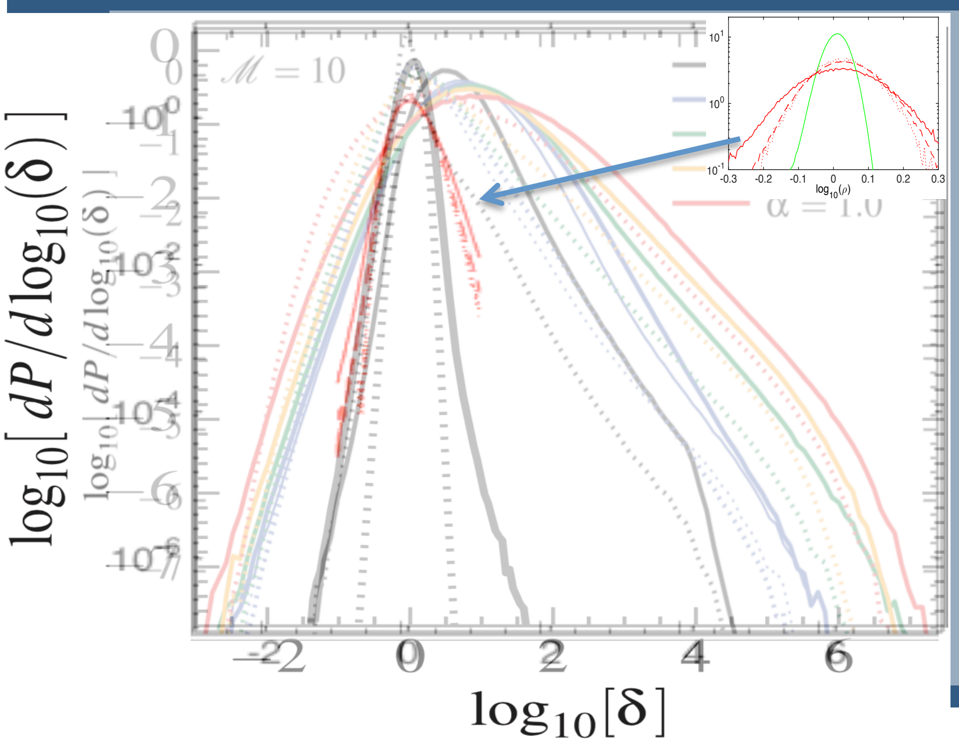


Hopkins & Lee 2016

Velocity Field tracers

Supersonic (M≅10) isothermal MHD turbulence (solenoidal driving), using GIZMO (MFM/MFV), in steady state

Still requires testing in the subsonic regime



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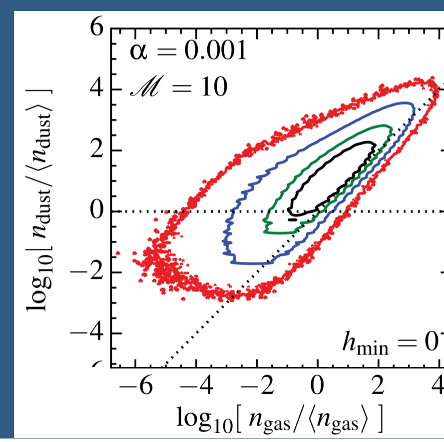
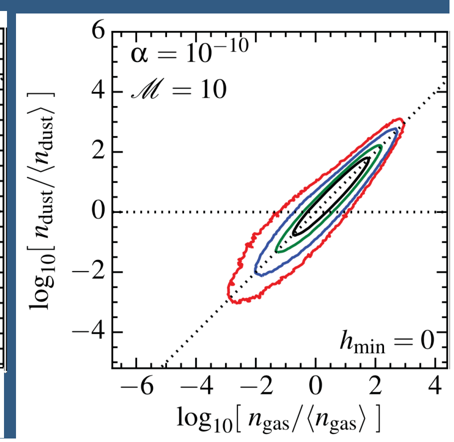


(Fully-coupled) Tracers



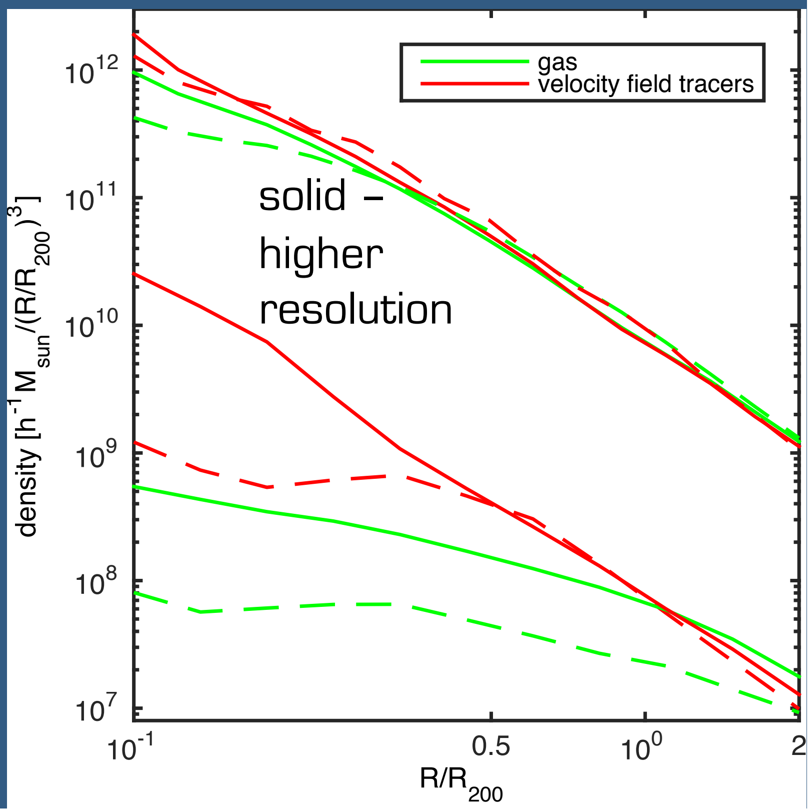
Genel et al. 2013

Velocity Field tracers

• In cosmological simulations, velocity field tracers disproportionally accumulate at halo centers, i.e. galaxies

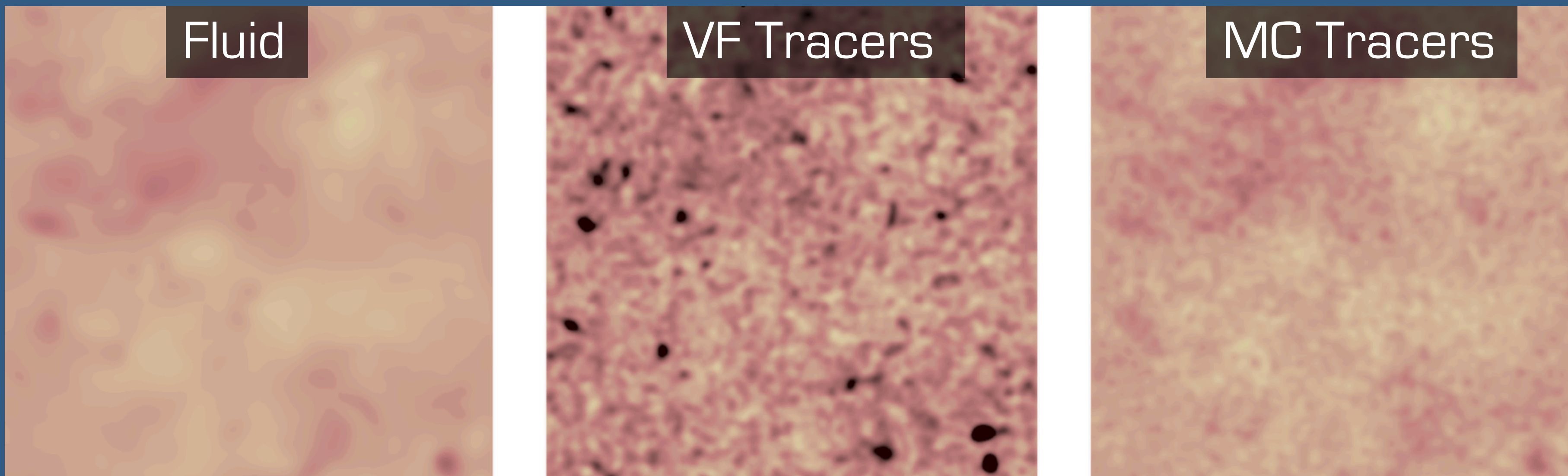
• No clear evidence for improvement at higher resolution, at least up to ~10^4 cells per halo

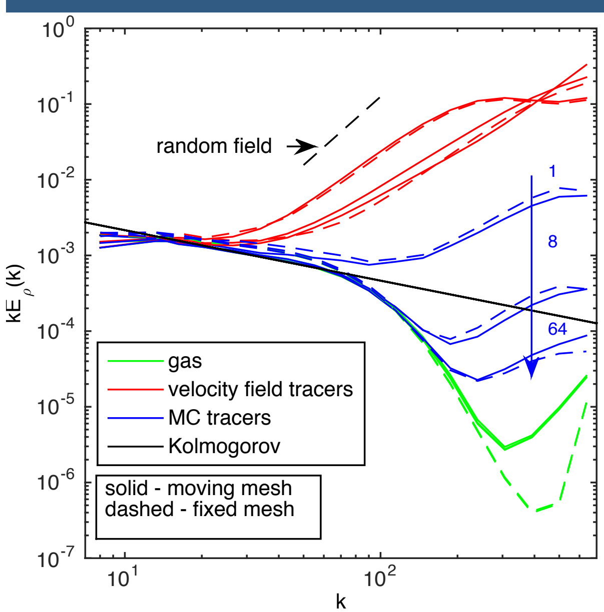
stacked density profiles



Genel et al. 2013

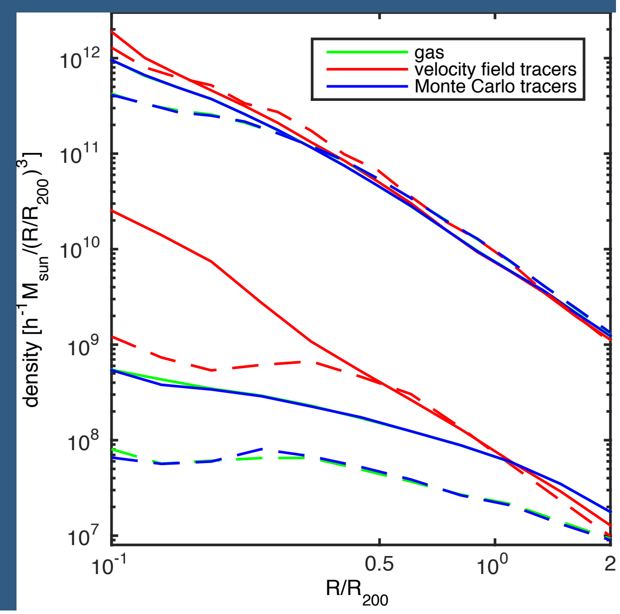
Tracer particles - comparison





Isothermal subsonic turbulence – density power spectrum

Cosmological halos – density profile

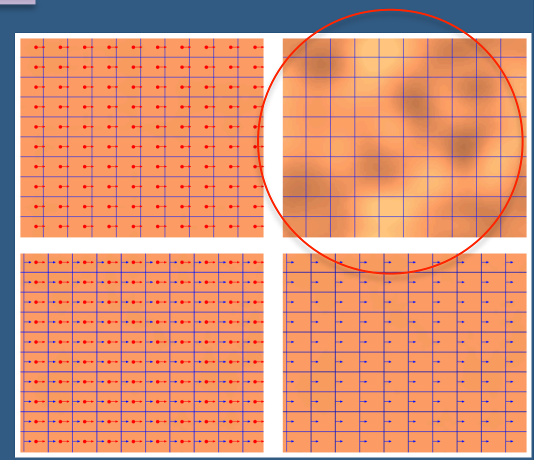


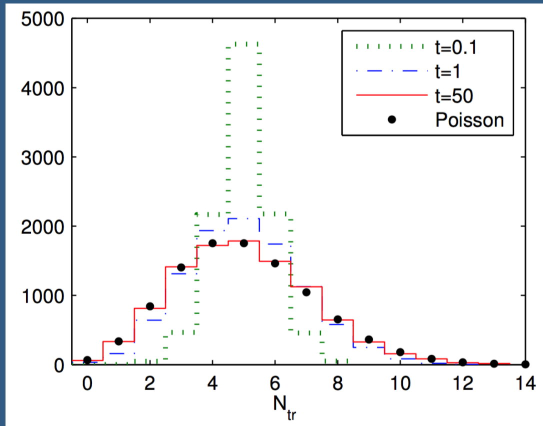
Mixed Monte Carlo tracers

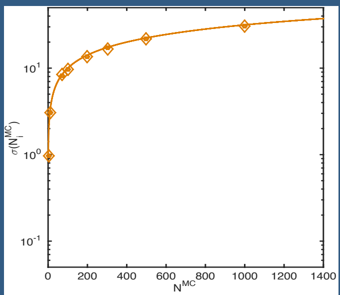
Genel et al. 2013
Genel et al. in prep.

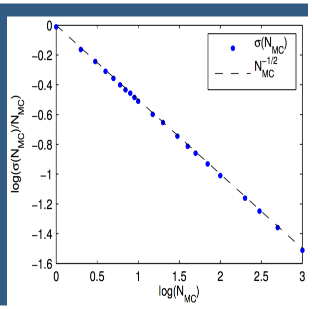
- Each cell can be approximated by an M/M/∞ queue, i.e.:
  - Arrivals are a Poisson process
  - Geometric ‘service time’ (Bernoulli trials until the first ‘success’=transfer)
  - All tracers receiving ‘service’
- The expected result: Poisson distribution for the number of tracers per cell
- The relative error can be brought down by increasing number of tracers per cell, but not so important...

$$\sigma(N_i^{\mathrm{MC}})|_{\mathrm{Mixed}} = \sqrt{N^{\mathrm{MC}}}$$









Genel et al. 2013
Genel et al. in prep.

Mixed Monte Carlo tracers

Numerical diffusion:

- Since the advection of MC tracers is the result of a series of Bernoulli trials, the dispersion in the distance travelled equals:

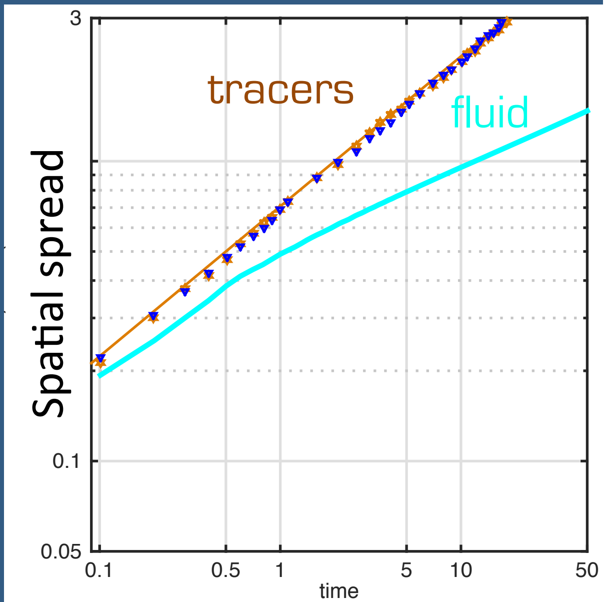
exchange probability

mean number of exchanges

$$\sigma(x_t - x_{t=0})|_{\mathrm{Mixed}} = \Delta x \sqrt{t/(\Delta x/v_x)(1-p)} = \sqrt{\Delta x D(1-p)},$$

cell size

total distance traveled



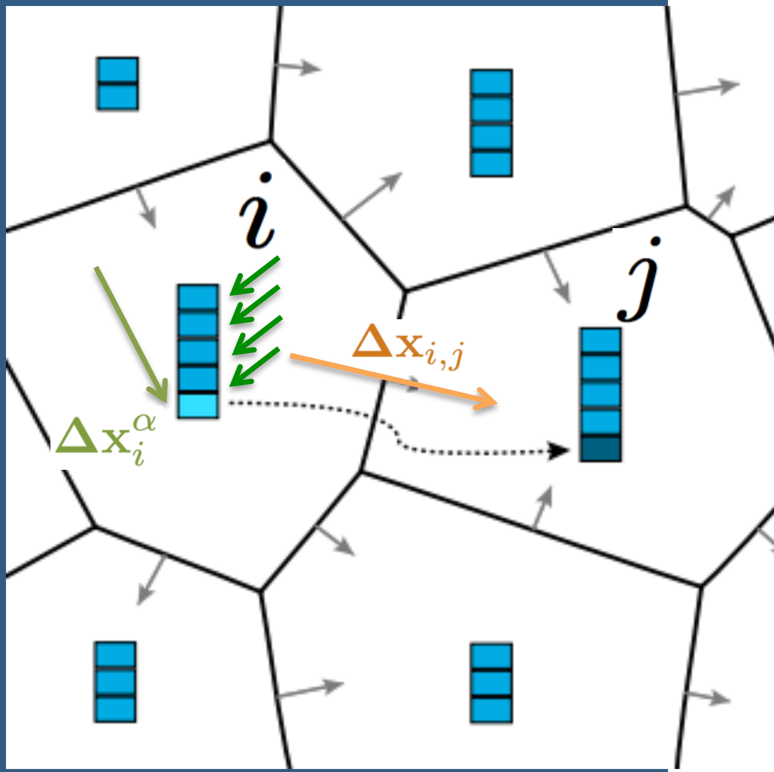
- But that is not how the fluid propagates

Monte Carlo tracers

Genel et al. 2013
Genel et al. in prep.

- Mixed MC tracers: every tracer gets an exchange probability of $p_{i,j}^{\rm flux} = \frac{\Delta M_{i,j}}{M_i}$

- Directional MC tracers: according to a score function ($S_{i,j}^{\alpha} = -\frac{\mathbf{\Delta x}_{i,j} \cdot \mathbf{\Delta x}_i^{\alpha}}{(t_{\rm now} - t_i^{\alpha})^{A_t}}$), only one tracer (the one 'at the front') is considered for exchange with a probability of $N_i^{\rm MC} \cdot p_{i,j}^{\rm flux}$



Monte Carlo tracers

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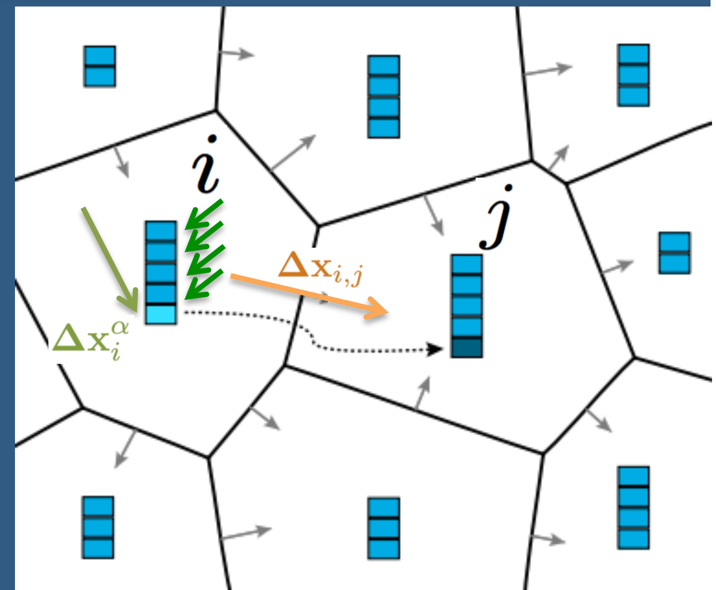

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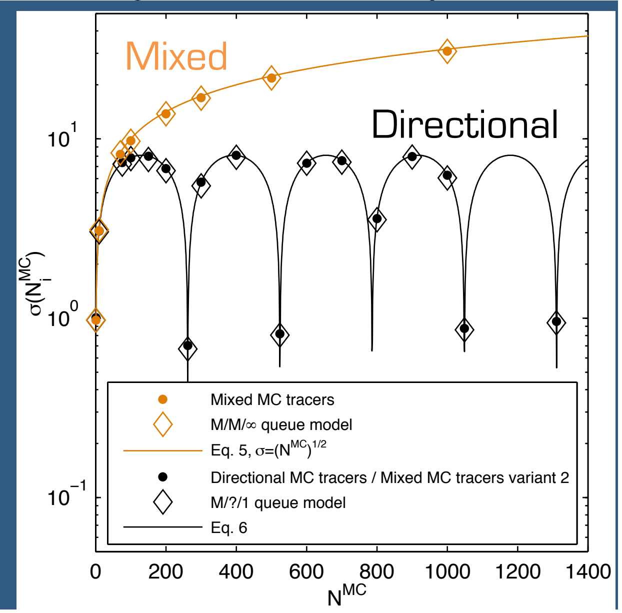
- With Directional MC tracers, the dispersion of the number of tracers per cell (per unit mass) – the 'Eulerian error' – does not monotonically grow with the adopted number of tracers



Monte Carlo tracers

Genel et al. 2013
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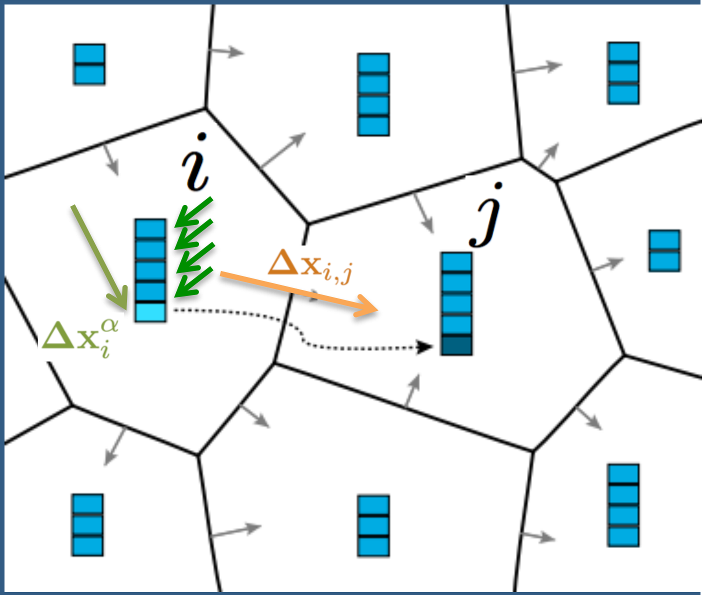
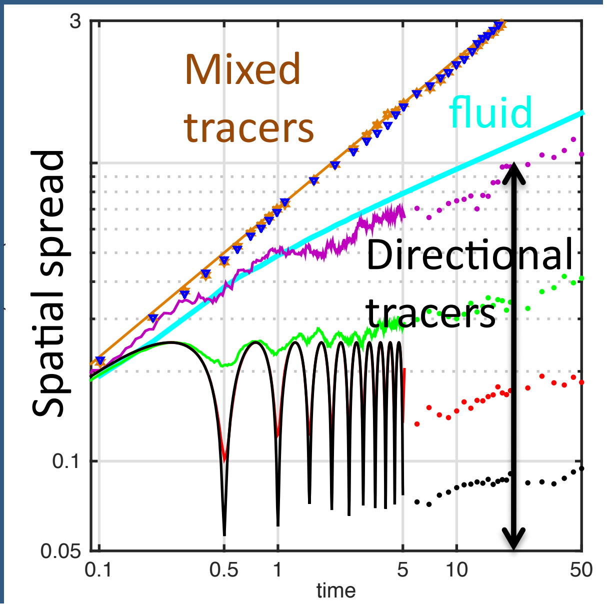
- With Directional MC tracers, the numerical diffusion is reduced, and grows less rapidly with time





Lagrangianity of numerical methods

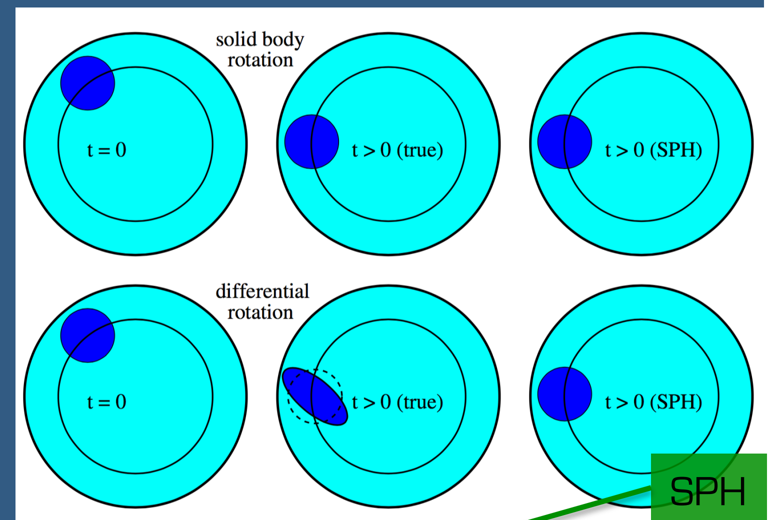
- What do we mean when we say a code is Lagrangian?
- Which conditions allow us to follow the native resolution elements of the code as Lagrangian elements?

numerical method	discretization type	element velocity matches fluid velocity	constant mass (no fluxes)	fluid deformations are properly followed	Lagrangian nature
Lagrangian[1]	mesh	$\surd$	$\surd$	$\surd$	Fully Lagrangian
Semi-Lagrangian[2]	mesh	$\times$	$\times$	$\surd$	Semi-Lagrangian
SPH	particle	$\surd$	$\surd$	$\times$	Pseudo-Lagrangian
'Lagrangian' MFM	particle	$\surd$	$\surd$	$\times$	Pseudo-Lagrangian
'Lagrangian' MFV	particle	$\surd$	$\times$	$\surd$	Quasi-Lagrangian
Moving-mesh/ALE[3]	mesh	$\approx$	$\times$	$\surd$	Quasi-Lagrangian
Eulerian/AMR	mesh	$\times$	$\times$	$\surd$	Non-Lagrangian

Lagrangianity of numerical methods

Vogelsberger et al. 2012:

"[...] the material initially in the SPH smoothing volume is forced to remain tied to this area and is not allowed to shear, inconsistent with the equations of motion of the real fluid."



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Lagrangianity of numerical methods

Hopkins 2015 (GIZMO code paper):
"[...] because we cannot perfectly follow the distortion of Lagrangian faces, the assumption made in the MFM method for the motion of the face in the Riemann problem [that the Lagrangian volume is distorting with the mean [...] motion of the volume partition, such that the mass on either 'side' of the state is conserved] will not *exactly* match the 'real' motion of the face calculated by directly time-differencing the positions estimated for it across two timesteps."

Or shortly in other words:
the 'real' motion of the volumes does not fully capture the distortion of the Lagrangian volumes, while the Riemann solver assumes it does.

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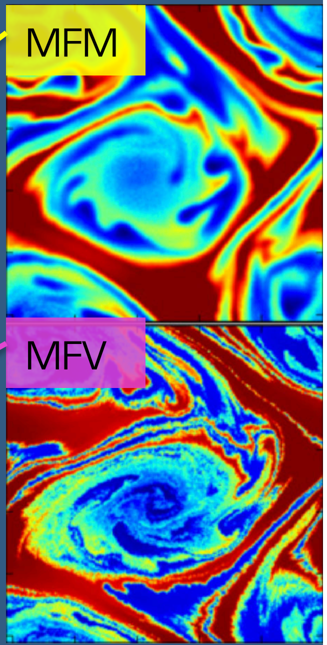
Lagrangianity of numerical methods

Hopkins 2015 (GIZMO code paper):

"[…] the second-order advection errors in the MFM method do not corrupt fluid mixing instabilities even in late-time, non-linear stages, where the true (physical) Lagrangian volumes of a fluid parcel would be distorted into arbitrarily complex shapes."



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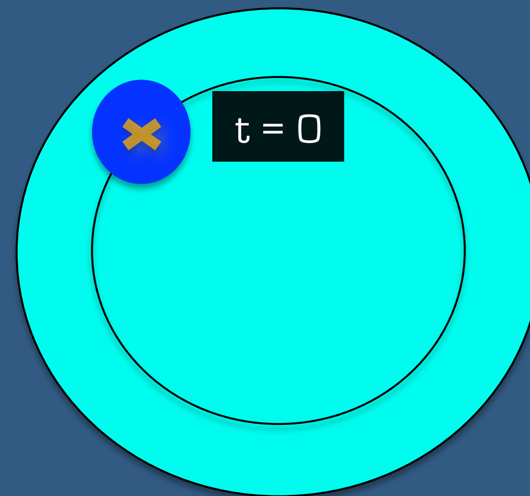
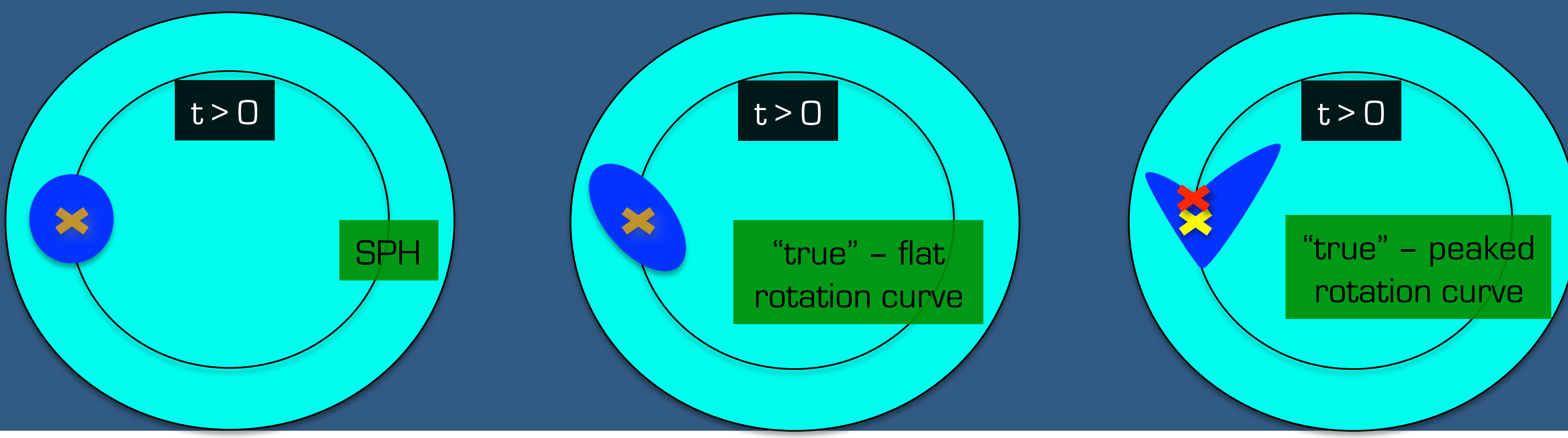
Moving-mesh/ALE[3]	mesh	≈	×	√	Quasi-Lagrangian
Eulerian/AMR	mesh	×	×	√	Non-Lagrangian

Lagrangianity of numerical methods

Going back to SPH (just because it is simple):

not only the ‘arbitrarily complicatedly distorted’ shape of the Lagrangian volume is not captured, but there is a first-order error on its center-of-mass



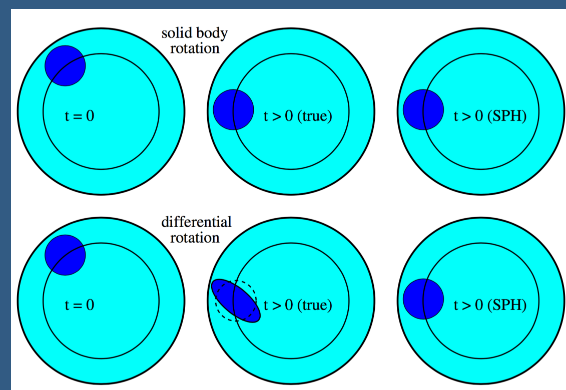


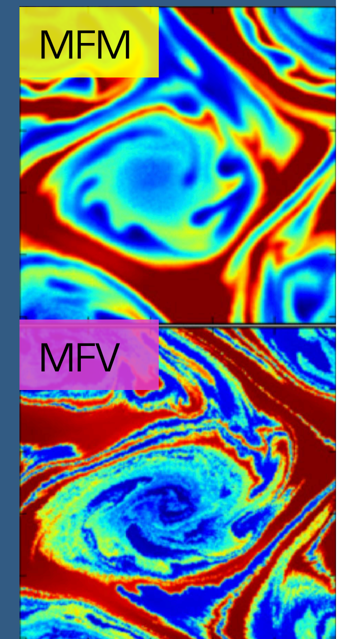
Lagrangianity of numerical methods

A curious side note:

the pseudo-Lagrangian nature of MFM causes it to enhance mixing, while

the pseudo-Lagrangian nature of SPH causes it to suppress mixing





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Conclusion

- Lagrangian analysis is underutilized in galaxy formation

- Velocity Field tracers: probably carries large errors for galaxy formation, requires careful use (at best)

- (Directional) Monte Carlo tracer particles: an independent approach, under development and study

- There exist no Lagrangian hydro codes – tracer particles are always necessary for a Lagrangian analysis